

Quantum Chromodynamics (QCD)

QCD: Currently the best known theory to describe the strong interaction.

SU(3) gauge theory with fermions in fundamental representation.

Fundamental degrees of freedom:

- gluons: A_{μ}^a , $a = 1, \dots, 8$
- quarks: ψ , $3(\text{color}) \times 4(\text{spin}) \times 6(\text{flavor})$ components

$$\mathcal{L}_{\text{QCD}} = \underbrace{-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}}_{\text{pure gauge part}} + \underbrace{\bar{\psi}(iD_{\mu}\gamma^{\mu} - m)\psi}_{\text{fermionic part}}$$

where

$$F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + gf^{abc}A_{\mu}^bA_{\nu}^c \quad \text{field strength}$$

$$D_{\mu} = \partial_{\mu} + gA_{\mu}^a \frac{\lambda^a}{2i} \quad \text{covariant derivative} \quad \longrightarrow \quad \text{gives quark–gluon interaction}$$

SU(3) group

SU(3): group of 3×3 unitary matrices with unit determinant:

$$U \in \text{SU}(3) \iff \begin{array}{l} \textcircled{1} \quad UU^\dagger = \mathbf{1}_{3 \times 3}, \quad \text{that is,} \quad U^{-1} = U^\dagger, \\ \textcircled{2} \quad \det U = 1. \end{array}$$

8 generators: Gell-Mann matrices λ^a ($a = 1, \dots, 8$)

Lie algebra of SU(3): Linear combinations $A = A^a \frac{\lambda^a}{2}$

$\textcircled{1}$ Hermitean: $A^\dagger = A,$

$\textcircled{2}$ traceless: $\text{Tr} A = 0.$

$$U = \exp(iA) = \exp\left(iA^a \frac{\lambda^a}{2}\right): \quad \text{elements of group SU(3).}$$

$$[A, B] = if^{abc} A^b B^c \frac{\lambda^a}{2}, \quad f^{abc}: \quad \text{structure coefficients.}$$

Quantum Chromodynamics (2)

\mathcal{L}_{QCD} is invariant under local gauge transformations:

$$A'_\mu(x) = G(x)A_\mu(x)G(x)^\dagger - \frac{i}{g} (\partial_\mu G(x)) G(x)^\dagger$$

$$\psi'(x) = G(x)\psi(x)$$

$$\bar{\psi}'(x) = \bar{\psi}(x)G^\dagger(x)$$

Only gauge invariant quantities are physical.

Properties of QCD:

- Asymptotic freedom:

Coupling constant $g \rightarrow 0$ when energy scale $\mu \rightarrow \infty$.

\implies Perturbation theory can be used at high energies.

- Confinement:

Coupling constant is large at low energies.

\implies Nonperturbative methods are required.

Quantum Chromodynamics (3)

Quantization using Feynman path integral:

$$\langle 0 | T[\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)] | 0 \rangle = \frac{\int [d\psi] [d\bar{\psi}] [dA_\mu] \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{iS[\psi, \bar{\psi}, A_\mu]}}{\int [d\psi] [d\bar{\psi}] [dA_\mu] e^{iS[\psi, \bar{\psi}, A_\mu]}}$$

e^{iS} oscillates \rightarrow hard to evaluate integrals.

Wick rotation: $t \rightarrow -it$ analytic continuation to Euclidean spacetime.

$\Rightarrow e^{iS} \rightarrow e^{-S_E}$, where

$$S_E = \int d^4x \mathcal{L}_E = \int d^4x \left[\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\psi} (D_\mu \gamma^\mu + m) \psi \right]$$

positive definite Euclidean action.

Quantum Chromodynamics (4)

Vector components: $\mu = 0, 1, 2, 3 \longrightarrow \mu = 1, 2, 3, 4$

Euclidean correlator

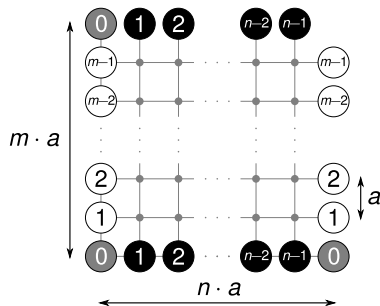
$$\langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | 0 \rangle_E = \frac{\int [d\psi] [d\bar{\psi}] [dA_\mu] \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) e^{-S_E[\psi, \bar{\psi}, A_\mu]}}{\int [d\psi] [d\bar{\psi}] [dA_\mu] e^{-S_E[\psi, \bar{\psi}, A_\mu]}}$$

Expectation value of $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$
with respect to positive definit measure $[d\psi] [d\bar{\psi}] [dA_\mu] e^{-S_E}$.

Lattice regularization

"Most systematic" nonperturbative approach:
lattice QFT

Take a finite segment of spacetime,
put fields at vertices of hypercubic lattice with lattice spacing a :



Usual boundary conditions:
Bosons:

Periodic in all directions

Fermions:

Time direction: antiperiodic

Space directions: periodic

Lattice regularization (2)

We have to discretize the action:

$$\begin{array}{ll} \text{integral over spacetime} & \int d^4x \longrightarrow \text{sum over sites} & a^4 \sum_x \\ \text{derivatives} & \partial_\mu \longrightarrow \text{finite differences} \end{array}$$

Momentum $p \leq \frac{\pi}{a} \implies$ natural UV cutoff.

At finite "a" results differ from the continuum value.

$$R^{\text{latt.}} = R^{\text{cont.}} + O(a^\nu)$$

for some dimensionless quantity R .

To get physical results, need to perform:

- 1 Infinite volume limit ($V \rightarrow \infty$),
- 2 Continuum limit ($a \rightarrow 0$).

Example: scalar field

Continuum action:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \lambda \phi^4$$

Simplest lattice action:

$$S = \sum_x a^4 \left(\frac{1}{2} \sum_{\mu=1}^4 \left[\frac{\phi_{x+\hat{\mu}} - \phi_{x-\hat{\mu}}}{2a} \right]^2 + \frac{1}{2} m^2 \phi_x^2 + \lambda \phi_x^4 \right)$$

where $\hat{\mu}$: unit vector in direction μ .

Introduce dimensionless quantities: $\phi' = \phi a$, $m' = ma$, $\lambda' = \lambda$

$$S = \sum_x \left(\phi_x'^2 \left[2 + \frac{1}{2} m'^2 \right] + \lambda' \phi_x'^4 - \frac{1}{2} \sum_{\mu=1}^4 \phi_{x+\hat{\mu}}' \phi_{x-\hat{\mu}}' \right)$$

Lattice spacing a does not appear explicitly in the calculations.

Example: scalar field

the ϕ^4 theory is trivial

it does not exist as an interactive continuum theory

it exists (as an interactive model) only as a cutoff theory

\implies the Higgs mass can not be larger than about 600 GeV

analytic & numerical works provide the evidence (Luscher, Weisz ...)

goes far beyond the perturbative unitarity arguments

in each point of the two dimensional parameter-space: m' and λ'

one can determine the renormalized mass and quartic coupling

high order hopping parameter expansion for $m_R a \gtrsim 0.5$

renormalization group equation techniques for $m_R a \lesssim 0.5$

(the latter approaches the infinite cutoff or continuum limit)

$m_R a = 0$ defines the critical line (continuum limit)

construct lines of constant physics (LCP):

$$a \rightarrow 0 \text{ but } m_R = \text{const.}, \lambda_R = \text{const.}$$

as "a" gets smaller along these LCPs the bare λ' gets larger
 actually before the LCPs reach the critical line one gets $\lambda' = \infty$
 (only the trivial theory $\lambda' = 0$ reaches the critical line)

assume that the maximum momenta are a few times larger than M_H
 maximal renormalized self-coupling, thus maximal Higgs mass

is obtained at the maximal bare coupling $\lambda' = \infty$
 using the Higgs vacuum expectation value (overall scale)

$$\text{one obtains } M_H \lesssim 600 \text{ GeV}$$

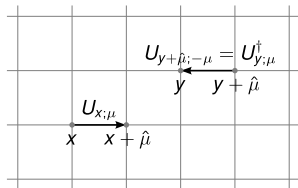
for even higher cutoffs (more than a few times) \Rightarrow smaller M_H

Yang–Mills theories on the lattice

Regularization has to maintain lattice version of gauge invariance.

Gauge fields \longrightarrow on links connecting neighboring sites.

- Continuum: A_μ , elements of Lie algebra of SU(3).
- Lattice: $U_\mu = e^{iagA_\mu}$, elements of group SU(3) itself.



$$U_{x+\hat{\mu};-\mu} = U_{x;\mu}^{-1} = U_{x;\mu}^\dagger$$

Lattice gauge transformation:

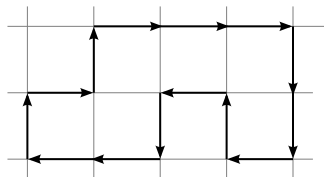
$$U'_{x;\mu} = G_x U_{x;\mu} G_{x+\hat{\mu}}^\dagger$$

$$\psi'_x = G_x \psi_x$$

$$\bar{\psi}'_x = \bar{\psi}_x G_x^\dagger$$

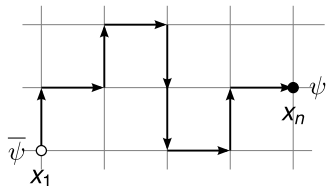
Gauge invariant quantities on the lattice

- Gluon loops



$$\text{Tr} [U_{x_1; \mu} U_{x_1 + \hat{\mu}; \nu} \cdots U_{x_1 - \hat{\epsilon}; \epsilon}]$$

- Gluon lines connecting q and \bar{q}



$$\bar{\psi}_{x_1} U_{x_1; \mu} U_{x_1 + \hat{\mu}; \nu} \cdots U_{x_n - \hat{\epsilon}; \epsilon} \psi_{x_n}$$

Gauge action

Continuum gauge action:

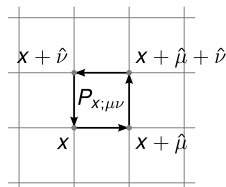
$$S_g^{\text{cont.}} = \int d^4x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$$

Simplest gauge invariant lattice action: Wilson action

$$S_g^{\text{Wilson}} = \beta \sum_x \sum_{\nu < \mu} \left(1 - \frac{1}{3} \text{Re} [P_{x;\mu\nu}] \right), \quad \beta = \frac{6}{g^2}, \quad S_g^{\text{latt.}} = S_g^{\text{cont.}} + O(a^2),$$

where $P_{x;\mu\nu}$ is the plaquette:

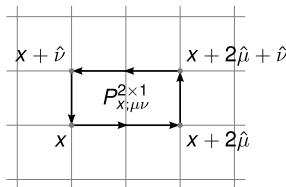
$$P_{x;\mu\nu} = \text{Tr} \left[U_{x;\mu} U_{x+\hat{\mu};\nu} U_{x+\hat{\nu};\mu}^\dagger U_{x;\nu}^\dagger \right]$$



Gauge action – Symanzik improvement

Add 2×1 gluon loops to Wilson action:

$$S_g^{\text{Symanzik}} = \beta \sum_x \sum_{\nu < \mu} \left\{ 1 - \frac{1}{3} (c_0 \text{Re}[P_{x;\mu\nu}] + c_1 \text{Re}[P_{x;\mu\nu}^{2 \times 1}] + c_1 \text{Re}[P_{x;\nu\mu}^{2 \times 1}]) \right\}$$



Consistency condition: $c_0 + 8c_1 = 1$.

$c_1 = -\frac{1}{12}$ gives tree level improvement $\implies S_g^{\text{latt.}} = S_g^{\text{cont.}} + O(a^4)$

Fermion doubling

Continuum fermion action

$$S_f = \int d^4x \bar{\psi}(\gamma^\mu \partial_\mu + m)\psi.$$

Naively discretized:

$$S_f^{\text{naive}} = a^4 \sum_x \left[\bar{\psi}_x \sum_{\mu=1}^4 \gamma_\mu \frac{\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}}{2a} + m \bar{\psi}_x \psi_x \right]$$

Inverse propagator:

$$G_{\text{naive}}^{-1}(p) = i\gamma_\mu \frac{\sin p_\mu a}{a} + m.$$

Extra zeros at $p_\mu = 0, \pm \frac{\pi}{a} \implies$ 16 zeros in 1st Brillouin zone.

In d dimensions 2^d fermions instead of 1 \implies fermion doubling.

Wilson fermions

$$S_f^W = S_f^{\text{naive}} - \underbrace{a \cdot \frac{r}{2} a^4 \sum_x \bar{\psi}_x \square \psi_x}_{\text{Wilson term}},$$

where

$$\square \psi_x = \sum_{\mu=1}^4 \frac{\psi_{x+\hat{\mu}} - 2\psi_x + \psi_{x-\hat{\mu}}}{a^2}.$$

$0 < r \leq 1$ Wilson parameter, usually $r = 1$.

$$G_W^{-1}(p) = G_{\text{naive}}^{-1}(p) + \frac{2r}{a} \sum_{\mu=1}^4 \sin^2(p_\mu a/2)$$

$m_{\text{doublers}} = O(a^{-1}) \implies$ doublers disappear in continuum limit.

Wilson fermions (2)

Work with dimensionless quantities: $a^{3/2}\psi \rightarrow \psi$

$$S_f^W = \sum_x \left\{ \bar{\psi}_x \sum_{\mu} [(\gamma_{\mu} - r) \psi_{x+\hat{\mu}} - (\gamma_{\mu} + r) \psi_{x-\hat{\mu}}] + (ma + 4r) \bar{\psi}_x \psi_x \right\}$$

Rescale ψ by $\sqrt{2\kappa}$, $\kappa = \frac{1}{2ma + 8r}$ hopping parameter.

Action including gauge fields:

$$S_f^W = \sum_x \left\{ \kappa \left[\sum_{\mu} \bar{\psi}_x (\gamma_{\mu} - r) U_{x;\mu} \psi_{x+\hat{\mu}} - \bar{\psi}_{x+\hat{\mu}} (\gamma_{\mu} + r) U_{x;\mu}^{\dagger} \psi_x \right] + \bar{\psi}_x \psi_x \right\}$$

Wilson fermions (3)

- Advantages

- 1 Kills all doublers.

- Disadvantages

- 1 No chiral symmetry at $a \neq 0$.

\implies Massless pions at $\kappa_c \neq \frac{1}{8r}$.

Additive quark mass renormalization.

- 2 Large discretization errors:

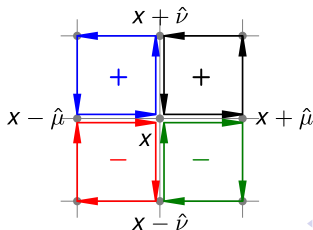
$$S_f^W = S_f^{\text{cont.}} + O(a)$$

Wilson fermions – Clover improvement

$$S_f^{\text{clover}} = S_f^{\text{W}} - \underbrace{\frac{ia_c\kappa r}{4} \sum_x \bar{\psi}_x \sigma_{\mu\nu} \mathcal{F}_{x;\mu\nu} \psi_x}_{\text{clover term}} = S_f^{\text{cont.}} + O(a^2), \quad \sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu]$$

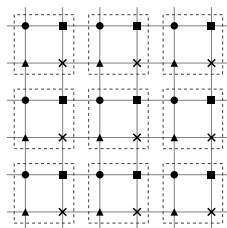
$$\mathcal{F}_{x;\mu\nu} = \frac{1}{4} \left(U_{x;\mu} U_{x+\hat{\mu};\nu} U_{x+\hat{\nu};\mu}^\dagger U_{x;\nu}^\dagger - U_{x-\hat{\nu};\nu}^\dagger U_{x-\hat{\mu}-\hat{\nu};\mu}^\dagger U_{x-\hat{\mu}-\hat{\nu};\nu} U_{x-\hat{\nu};\nu} + \right. \\ \left. + U_{x;\nu} U_{x-\hat{\mu}+\hat{\nu};\mu}^\dagger U_{x-\hat{\mu};\nu}^\dagger U_{x-\hat{\mu};\mu} - U_{x;\mu} U_{x+\hat{\mu}-\hat{\nu};\nu}^\dagger U_{x-\hat{\nu};\mu}^\dagger U_{x-\hat{\nu};\nu} \right)$$

discretized version of field strength $F_{\mu\nu}$.



Kogut–Susskind (staggered) fermions

Fermion degrees of freedom \longrightarrow corners of hypercube.



In d dimensions:

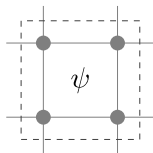
$2^{d/2}$ spinor components of Dirac spinors

2^d corners of hypercube

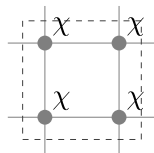
\implies describes $2^d / 2^{d/2} = 2^{d/2}$ flavors (tastes).

If $d = 4 \implies 4$ flavors (tastes) $\implies 4^{\text{th}}$ rooting required.

Kogut-Susskind (staggered) fermions (2)



3(color) \times 4(spin) components



3(color) \times 1(spin) components

$$S_f^S = \sum_x \bar{\chi}_x \left\{ \frac{1}{2} \sum_{\mu} \eta_{x,\mu} \left(U_{x;\mu} \chi_{x+\hat{\mu}} - U_{x-\hat{\mu};\mu}^\dagger \chi_{x-\hat{\mu}} \right) + m a \chi_x \right\},$$

where

$$\eta_{x,\mu} = (-1)^{\sum_{\nu=1}^{\mu-1} x_{\nu}}$$

staggered phase.

Kogut–Susskind (staggered) fermions (3)

- Advantages

- 1 Remnant chiral symmetry at $a \neq 0$

⇒ no additive quark mass renormalization.

- 2 $O(a^2)$ discretization errors.

- 3 Fast.

- Disadvantages

- 1 4 tastes (flavors) instead of 1

⇒ rooting trick required.

- 2 Taste symmetry breaking.

Integral over fermions

Full lattice QCD action

$$S(U, \psi, \bar{\psi}) = \underbrace{S_g(U)}_{\text{gluonic part}} - \underbrace{\bar{\psi} M(U) \psi}_{\text{fermionic part}}$$

Fermions are described by Grassmann variables \longrightarrow have to integrate out analytically.

$$\int [dU] [d\bar{\psi}] [d\psi] e^{-S_g(U) + \bar{\psi} M(U) \psi} = \int [dU] e^{-S_g(U)} \det M(U)$$

\implies Effective action for gluons

$$S_{\text{eff.}}(U) = S_g(U) - \ln(\det M(U)).$$

Staggered fermion matrix describes 4 tastes.

Rooting trick: for n_f flavors, take power $\frac{n_f}{4}$ of determinant:

$$S_{\text{eff.}}^S(U) = S_g(U) - \ln(\det M(U)^{n_f/4}) = S_g(U) - \frac{n_f}{4} \ln(\det M(U))$$

Expectation values of fermionic quantities

$$\mathcal{O}(x, y) = \left(\bar{\psi}^u \psi^d \right)_y \left(\bar{\psi}^d \psi^u \right)_x \quad \text{fermionic operator}$$

$$\begin{aligned} \langle 0 | \mathcal{O}(x, y) | 0 \rangle &= \frac{\int [dU] [d\bar{\psi}] [d\psi] \bar{\psi}_y^{u,a} \psi_y^{d,a} \bar{\psi}_x^{d,b} \psi_x^{u,b} e^{-S_g(U) + \bar{\psi} M(U) \psi}}{\int [dU] [d\bar{\psi}] [d\psi] e^{-S_g(U) + \bar{\psi} M(U) \psi}} = \\ &= \frac{\int [dU] \left[M_{x,y}^{-1,u}(U) \right]^{ab} \left[M_{y,x}^{-1,d}(U) \right]^{ba} \det M(U) e^{-S_g(U)}}{\int [dU] \det M(U) e^{-S_g(U)}} = \\ &= \frac{\int [dU] \text{Tr}_{\text{color,spin}} \left[\left(M_{x,y}^{-1,u} \right) \left(M_{y,x}^{-1,d} \right) \right] e^{-S_{\text{eff.}}(U)}}{\int [dU] e^{-S_{\text{eff.}}(U)}}. \end{aligned}$$

Expectation values of fermionic quantities (2)

Expectation value of
with respect to action

$$\mathcal{O} = \left(\bar{\psi}^u \psi^d \right)_y \left(\bar{\psi}^d \psi^u \right)_x$$

$$S(U, \psi, \bar{\psi}) = S_g(U) - \bar{\psi} M(U) \psi.$$

↓

Expectation value of
with respect to action

$$\mathcal{O}' = \text{Tr}_{\text{color,spin}} \left[\left(M_{x,y}^{-1,u} \right) \left(M_{y,x}^{-1,d} \right) \right]$$

$$S_{\text{eff.}}(U) = S_g(U) - \ln(\det M(U)).$$

$$\langle 0 | \mathcal{O} | 0 \rangle = \frac{\int [dU] [d\bar{\psi}] [d\psi] \mathcal{O} e^{-S(U, \psi, \bar{\psi})}}{\int [dU] [d\bar{\psi}] [d\psi] e^{-S(U, \psi, \bar{\psi})}} = \frac{\int [dU] \mathcal{O}' e^{-S_{\text{eff.}}(U)}}{\int [dU] e^{-S_{\text{eff.}}(U)}}$$

Importance sampling

Monte Carlo simulation: calculate $\langle 0 | \mathcal{O} | 0 \rangle$ stochastically.

Naive way: take random gauge configurations U_α according to the uniform distribution and calculate the weighed average:

$$\langle 0 | \mathcal{O} | 0 \rangle = \frac{\sum_{\alpha} \mathcal{O}_{\alpha} e^{-S_{\alpha}}}{\sum_{\alpha} e^{-S_{\alpha}}} \quad \begin{array}{l} S_{\alpha}: \text{value of } S_{\text{eff.}} \text{ at } U_{\alpha}, \\ \mathcal{O}_{\alpha}: \text{value of } \mathcal{O} \text{ at } U_{\alpha}. \end{array}$$

S_{α} large for most configurations \longrightarrow small portion of configurations give significant contribution.

Importance sampling: generate configurations with probability based on their importance \longrightarrow probability of U_{α} is proportional to $e^{-S_{\alpha}}$.

Then $\langle 0 | \mathcal{O} | 0 \rangle = \frac{1}{N} \sum_{\alpha=1}^N \mathcal{O}_{\alpha}$ with relative error $\frac{1}{\sqrt{N}}$.

Importance sampling (2)

Simplest method: Metropolis algorithm.

Choose an initial configuration U_0 .

- 1 Generate U_{k+1} from U_k with a small random change.
 - 2 Measure the change ΔS in the action.
 - 3 If $\Delta S \leq 0$, keep U_{k+1} .
 - 4 If $\Delta S > 0$, keep U_{k+1} with a probability of $e^{-\Delta S}$.
- U_0 is far from the region where e^{-S} is significant.
 \implies Many steps required to reach equilibrium distribution:
Thermalization time.
 - $U_k \longrightarrow U_{k+1}$ by small change.
 \implies Subsequent configurations are not independent.
Number of steps required to reach next independent configuration: Autocorrelation time.

Setting the scale

All quantities in the calculation are in lattice units
 → lattice spacing a has to be determined.

Process of obtaining a :

- 1 Choose physical quantity A such that
 - experimental value $A_{\text{exp.}}$ is well known,
 - easily measurable on the lattice,
 - not sensitive to discretization errors,
 - $[A] = (\text{GeV})^\nu$, $\nu \neq 0$.
- 2 Measure dimensionless $A'_{\text{latt.}} = A_{\text{latt.}} \cdot a^\nu$ on the lattice.
- 3 Setting $A_{\text{latt.}} = A_{\text{exp.}}$ yields $a = \left(\frac{A'_{\text{latt.}}}{A_{\text{exp.}}} \right)^{1/\nu}$.

Setting the scale (2)

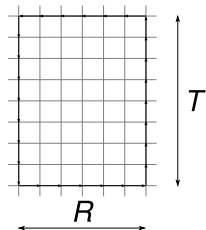
- 1 $A = \sigma$ string tension

$$\sigma = \lim_{R \rightarrow \infty} \frac{dV(R)}{dR}$$

Experimental value: $\sqrt{\sigma} = 465 \text{ MeV}$

$$V(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \ln[W(R, T)],$$

$$W(R, T) =$$



Static $q\bar{q}$ potential

Setting the scale (3)

- 2 $A = r_0$ Sommer parameter,

$$R^2 \cdot \left. \frac{dV(R)}{dR} \right|_{R=r_0} = 1.65$$

Experimental value: $r_0 = 0.469(7)$ fm

- 3 $A = F_K$ leptonic decay constant of Kaon
Experimental value: $f_K = 159.8$ MeV

Hadron spectroscopy

Aim: obtain mass of hadron state \mathcal{O} .

$\langle 0 | \mathcal{O}(t) \bar{\mathcal{O}}(0) | 0 \rangle$ amplitude of process:
hadron state created by $\bar{\mathcal{O}}$ at instant 0 is annihilated at instant t .

Project to $p = 0$:

$$\mathcal{O}^{(p=0)}(t) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} \mathcal{O}(t, \mathbf{x}) \Big|_{\mathbf{p}=0} = \sum_{\mathbf{x}} \mathcal{O}(t, \mathbf{x})$$

Obtaining operator with given parity:

$$\mathcal{O}_{\pm} = \frac{1}{2} \left(\mathcal{O} \pm P \mathcal{O} P^{-1} \right)$$

Hadron spectroscopy (2)

Euclidean time evolution:

$$\mathcal{O}(t) = e^{Ht} \mathcal{O}(0) e^{-Ht}$$

H : Hamiltonian of QCD.

$$\langle 0 | \mathcal{O}(t) \bar{\mathcal{O}}(0) | 0 \rangle = \sum_i | \langle i | \bar{\mathcal{O}}(0) | 0 \rangle |^2 \cdot e^{-(E_i - E_0)t}$$

E_i : energy of eigenstate $|i\rangle$,

E_0 : energy of vacuum state,

$m_i = E_i - E_0$: mass of hadronic state $|i\rangle$.

At large t

$$\langle 0 | \mathcal{O}(t) \bar{\mathcal{O}}(0) | 0 \rangle \approx | \langle l | \bar{\mathcal{O}}(0) | 0 \rangle |^2 \cdot e^{-m_l t}$$

$|l\rangle$: lowest state with $\langle l | \bar{\mathcal{O}}(0) | 0 \rangle \neq 0$.

Finite temperature QFT

Partition function of a quantum system with Hamiltonian H at temperature T :

$$Z = \text{Tr} \left[e^{-H/T} \right]$$

Bosonic QFT \longrightarrow integral over 3d field configurations φ .

$$\text{Tr} \left[e^{-H/T} \right] = \int [d\varphi] \langle \varphi | e^{-H/T} | \varphi \rangle$$

Integrand: $|\varphi\rangle \rightarrow |\varphi\rangle$ transition probability amplitude after imaginary time $t = -i/T$.

$$\langle \varphi | e^{-H/T} | \varphi \rangle = \int_{\substack{\phi|_{t=0}=\varphi \\ \phi|_{t=-i/T}=\varphi}} [d\phi] \exp \left[i \int_{t=0}^{t=-i/T} dt L \left(\phi, \frac{\partial \phi}{\partial it} \right) \right]$$

Finite temperature QFT (2)

$$\langle \varphi | e^{-H/T} | \varphi \rangle = \int_{\substack{\phi|_{t=0}=\varphi \\ \phi|_{t=-i/T}=\varphi}} [d\phi] \exp \left[i \int_{t=0}^{t=-i/T} dt \int d^3\mathbf{x} \mathcal{L} \left(\phi, \frac{\partial\phi}{\partial it} \right) \right]$$

Introduce $\tau = it \rightarrow$ Euclidean Lagrangian appears

$$= \int_{\substack{\phi|_{\tau=0}=\varphi \\ \phi|_{\tau=1/T}=\varphi}} [d\phi] \exp \left[- \int_{\tau=0}^{\tau=1/T} d\tau \int d^3\mathbf{x} \mathcal{L}_E \left(\phi, \frac{\partial\phi}{\partial\tau} \right) \right]$$

$$Z = \text{Tr} \left[e^{-H/T} \right] = \int [d\varphi] \int_{\substack{\phi|_{\tau=0}=\varphi \\ \phi|_{\tau=1/T}=\varphi}} [d\phi] \exp \left[- \int_{\tau=0}^{\tau=1/T} d\tau \int d^3\mathbf{x} \mathcal{L}_E \left(\phi, \frac{\partial\phi}{\partial\tau} \right) \right]$$

$$= \int_{\phi|_{\tau=0}=\phi|_{\tau=1/T}} [d\phi] e^{-S_E(\phi)} \rightarrow \text{Euclidean path integral over 4d field configurations with periodicity } 1/T.$$

Finite temperature QCD

$$\begin{aligned}
 Z_{\text{QCD}} &= \int [dU] [d\bar{\psi}] [d\psi] e^{-S_E(U, \psi, \bar{\psi})} \\
 &= \int [dU] [d\bar{\psi}] [d\psi] \exp \left[\int_0^{1/T} dx_4 \int d^3\mathbf{x} \mathcal{L}_E(U, \psi, \bar{\psi}) \right]
 \end{aligned}$$

Boundary condition in the imaginary time (temperature) direction:

Gluons: periodic,

Quarks: antiperiodic.

Temperature of lattice with temporal extension N_t :

$$T = \frac{1}{a \cdot N_t}.$$

Increase of β $\xrightarrow{\text{asymptotic freedom}}$ decrease of a \implies increase of T .