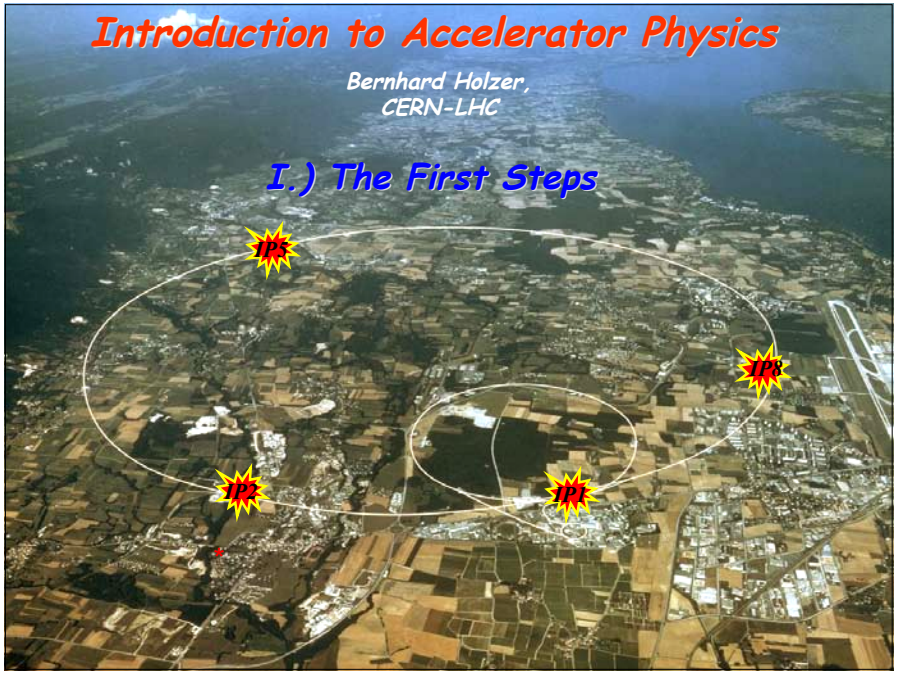

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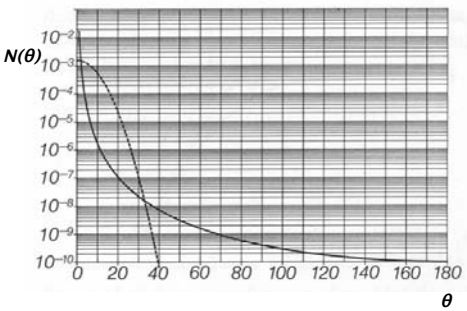
A Bit of History

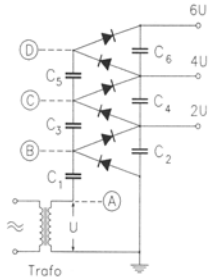
$$N(\theta) = \frac{N_i n t Z^2 e^4}{(8\pi\varepsilon_0)^2 r^2 K^2} * \frac{1}{\sin^4(\theta/2)}$$



Rutherford Scattering, 1906

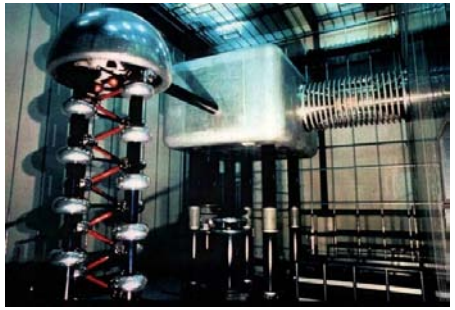
Using radioactive particle sources:
α-particles of some MeV energy

1.) Electrostatic Machines: The Cockcroft-Walton Generator



1928: Encouraged by Rutherford Cockcroft and Walton start the design & construction of a high voltage generator to accelerate a proton beam

1932: First particle beam (protons) produced for nuclear reactions: splitting of Li-nuclei with a proton beam of 400 keV



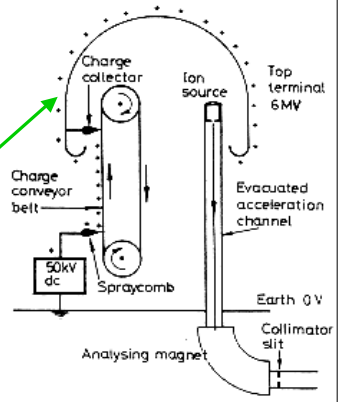
Particle source: Hydrogen discharge tube on 400 kV level
Accelerator: evacuated glas tube
Target: Li-Foil on earth potential

Technically: rectifier circuit, built of capacitors and diodes (Greinacher)

robust, simple, on-knob machines
largely used in history as pre-accelerators for proton and ion beams
recently replaced by modern structures (RFQ)

2.) Electrostatic Machines:
(Tandem –) van de Graaff Accelerator (1930 ...)

creating high voltages by mechanical transport of charges



** Terminal Potential: U ≈ 12 ...28 MV*
using high pressure gas to suppress discharge (SF_6)

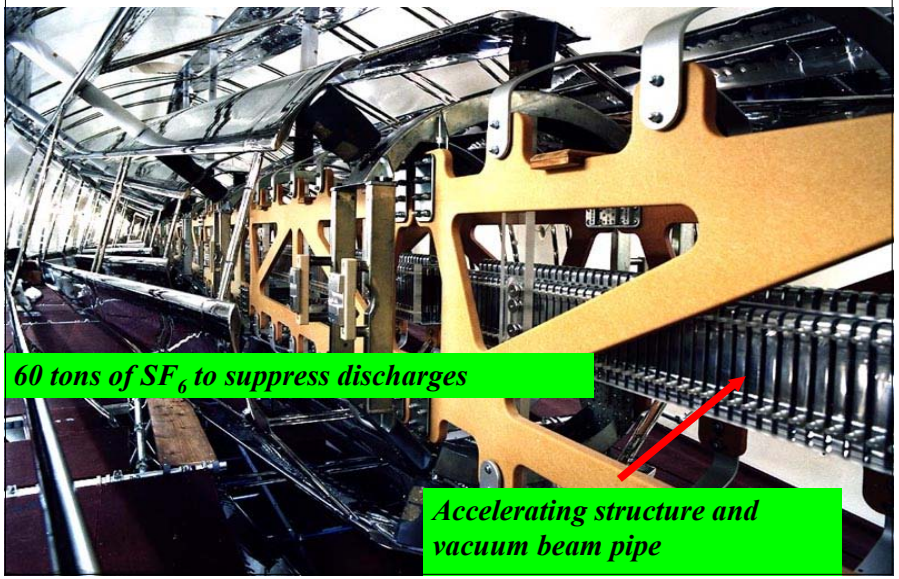
*Problems: * Particle energy limited by high voltage discharges*
** high voltage can only be applied once per particle ...*
... or twice ?

The „Tandem principle“: Apply the accelerating voltage twice ...
... by working with negative ions (e.g. H^-) and stripping the electrons in the centre of the structure

Example for such a „steam engine“: 12 MV-Tandem van de Graaff Accelerator at MPI Heidelberg

... and how it looks inside

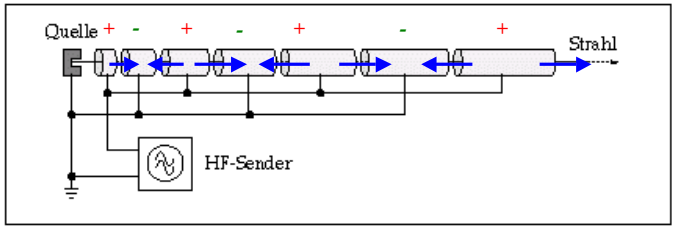
"Vivitron" Strassbourg



3.) The first RF-Accelerator: „Linac"

1928, Wideroe: ***how can the acceleration voltage be applied several times to the particle beam***

schematic Layout:



Energy gained after n acceleration gaps

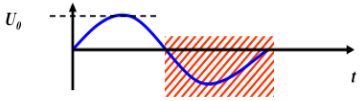
$$E_n = n * q * U_0 * \sin\psi_s$$

n *number of gaps between the drift tubes*
q *charge of the particle*
U_0 *Peak voltage of the RF System*
Ψ_S *synchronous phase of the particle*

** acceleration* ***of the proton in the*** *first gap*
** voltage has to be „flipped"* ***to get the right sign in the second gap →*** *RF voltage*
→ shield the particle in drift tubes ***during the negative half wave of the RF voltage***

Wideroe-Structure: the drift tubes

shielding of the particles during the negative half wave of the RF



Time span of the negative half wave: $\tau_{RF}/2$

Length of the Drift Tube: $l_i = v_i * \frac{\tau_{rf}}{2}$

Kinetic Energy of the Particles $E_i = \frac{1}{2} m v^2$

$$\rightarrow v_i = \sqrt{2E_i/m}$$

$$l_i = \frac{1}{\nu_{rf}} * \sqrt{\frac{i * q * U_{0 * \sin\psi_s}}{2m}}$$

valid for non relativistic particles ...

Alvarez-Structure: 1946, surround the whole structure by a rf vessel

Energy: ≈ 20 MeV per Nucleon β ≈ 0.04 … 0.6, Particles: Protons/Ions

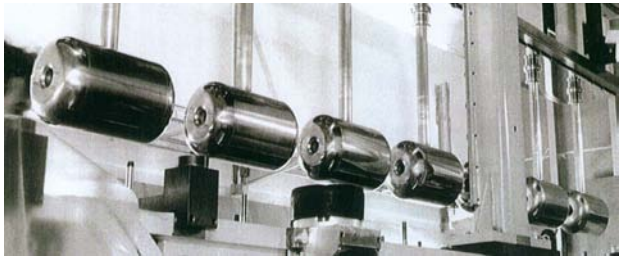
Example: DESY Accelerating structure of the Proton Linac

E_{total} = 988 MeV

m_0c^2 = 938 MeV

p = 4 MeV/c

E_{kin} = 50 MeV



Beam energies

1.) reminder of some relativistic formula

rest energy $E_0 = m_0 c^2$

total energy $E = \gamma * E_0 = \gamma * m_0 c^2$

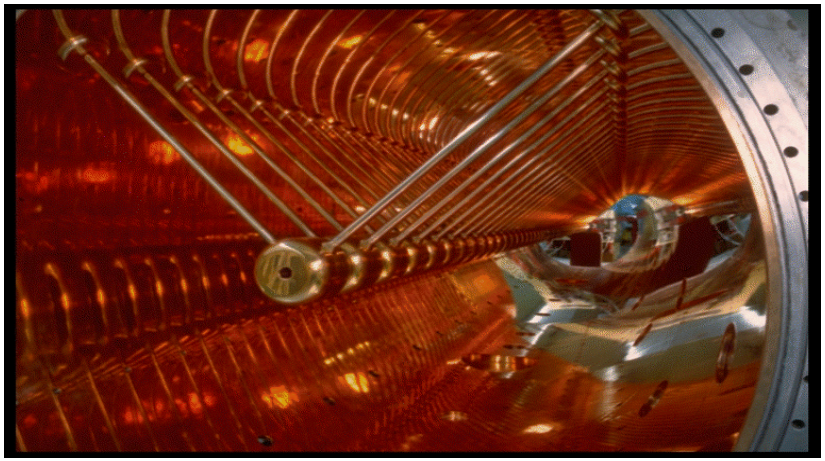
momentum $E^2 = p^2c^2 + m_0^2c^4$

kinetic energy $E_{kin} = E_{total} - m_0 c^2$

GSI: Unilac, typical Energie ≈ 20 MeV per Nukleon, β ≈ 0.04 … 0.6, Protons/Ions, ν = 110 MHz

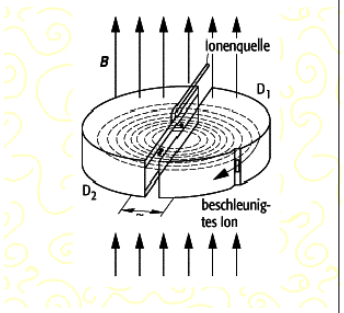
Energy Gain per „Gap“:

$$W = q\,U_0 \sin \omega_{RF} t$$



Application: until today THE standard proton / ion pre-accelerator
CERN Linac 4 is being built at the moment

4.) The Cyclotron: (Livingston / Lawrence ~1930)



Idea: Bend a Linac on a Spiral
Application of a constant magnetic field
keep B = const, RF = const

➔ *Lorentzforce*

$$\vec{F} = q * (\vec{v} \times \vec{B}) = q * v * B$$

circular orbit

$$q * v * B = \frac{m * v^2}{R} \quad \rightarrow \quad B * R = p / q$$

increasing radius for
increasing momentum
➔ *Spiral Trajectory*

revolution frequency

$$\omega_z = \frac{q}{m} * B_z$$

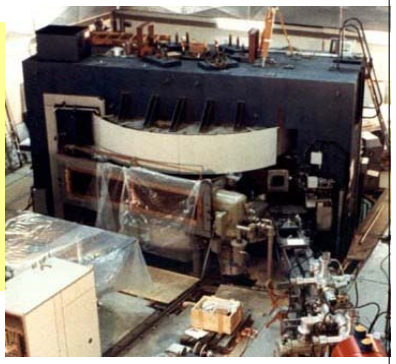
the cyclotron (rf-) frequency
is independent of the momentum

Cyclotron:

! ω is constant for a given q & B

!! B*R = p/qlarge momentum → huge magnets

!!!! $\omega \sim 1/m \neq$ const. works properly only for non relativistic particles



PSI Zurich

Application:

Work horses for medium energy protons

Proton / Ion Acceleration up to ≈ 60 MeV (proton energy)

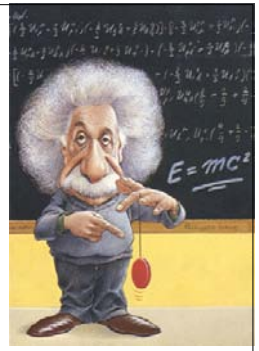
nuclear physics

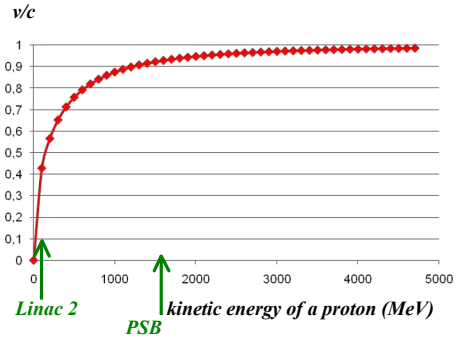
radio isotope production, proton / ion therapy

Beam Energy

... so sorry, here we need help from Albert:

$$\gamma = \frac{E_{total}}{mc^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \frac{v}{c} = \sqrt{1-\frac{mc^2}{E^2}}$$





CERN Accelerators	kin. Energy	γ
Linac 2	60 MeV	1.06
PS	26 GeV	27
SPS	450 GeV	480
LHC	7 TeV	7460

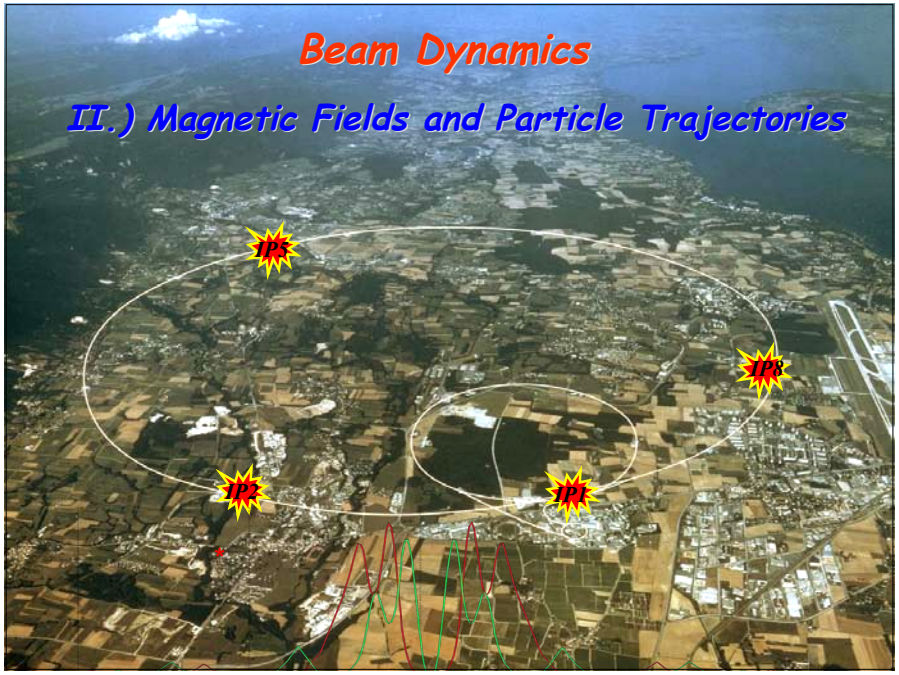
remember: proton mass = 938 MeV

5.) Synchrotrons / Storage Rings / Colliders:

Wideroe 1943, McMillan, Veksler 1944,
Courant, Livingston, Snyder 1952

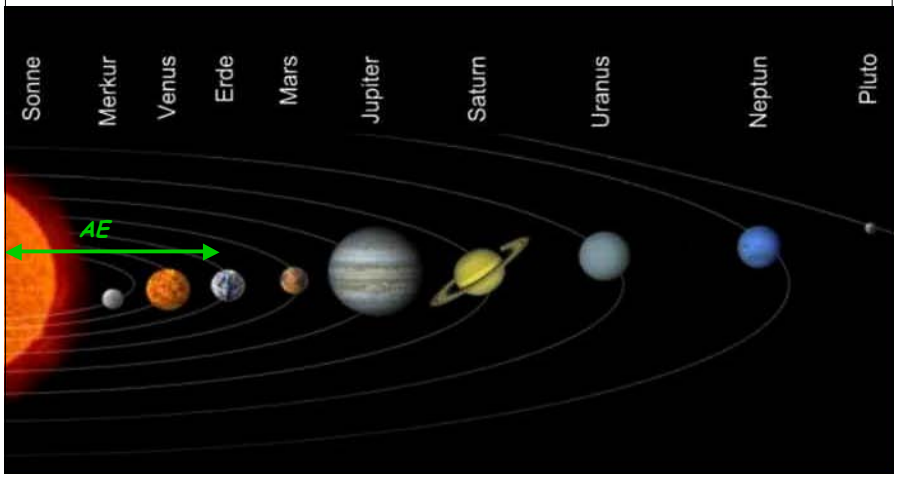
Idea: define a circular orbit of the particles,
keep the beam there during acceleration,
put magnets at this orbit to guide and focus

Advanced Photon Source,
Berkley



Largest storage ring: *The Solar System*

astronomical unit: *average distance earth-sun*
*1AE ≈ 150 *10⁶ km*
Distance Pluto-Sun ≈ 40 AE



Luminosity Run of a typical storage ring:

LHC Storage Ring: *Protons accelerated and stored for 12 hours*
distance of particles travelling at about v ≈ c
$L = 10^{10}$-10^{11} km
... several times Sun - Pluto and back

→ *guide the particles on a well defined orbit („design orbit")*
→ *focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.*

1.) Introduction and Basic Ideas

„ ... in the end and after all it should be a kind of circular machine“

→ *need transverse deflecting force*

Lorentz force $\vec{F} = q*(\cancel{\vec{E}} + \vec{v} \times \vec{B})$

typical velocity in high energy machines: $v \approx c \approx 3*10^8\ m/s$

Example:

$B = 1\,T \quad \rightarrow \quad F = q*3*10^8 \frac{m}{s} * 1 \frac{Vs}{m^2}$

$F = q*\underbrace{300 \frac{MV}{m}}_{E}$

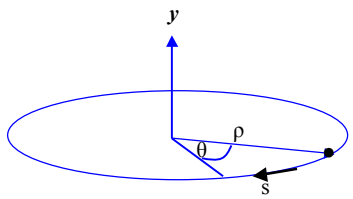
equivalent el. field ... E

technical limit for el. field:

$E \leq 1 \frac{MV}{m}$

old greek dictum of wisdom:
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

The ideal circular orbit



circular coordinate system

condition for circular orbit:

Lorentz force $F_L = e\,v\,B$

centrifugal force $F_{centr} = \frac{\gamma\, m_0\, v^2}{\rho}$

$\frac{\gamma\, m_0\, v^{\cancel{2}}}{\rho} = e\,\cancel{v}\,B$

$\frac{p}{e} = B\,\rho$

$B\,\rho =$ *"beam rigidity"*

2.) The Magnetic Guide Field

Dipole Magnets:

define the ideal orbit
homogeneous field created
by two flat pole shoes

$$B = \frac{\mu_0 \, n \, I}{h}$$

Normalise magnetic field to momentum:

$$\frac{p}{e} = B\rho \longrightarrow \frac{1}{\rho} = \frac{e\,B}{p}$$

convenient units:

$$B = [T] = \left[\frac{Vs}{m^2}\right] \qquad p = \left[\frac{GeV}{c}\right]$$

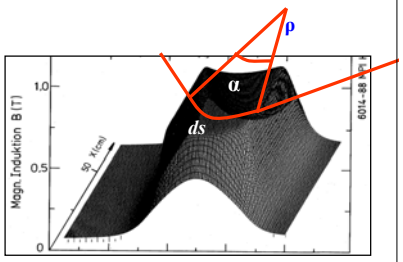
Example LHC:

$$\left.\begin{aligned} B &= 8.3\,T \\ p &= 7000\,\frac{GeV}{c} \end{aligned}\right\}$$

$$\frac{1}{\rho} = e\,\frac{8.3\ Vs/m^2}{7000*10^9\ eV/c} = \frac{8.3\,s*3*10^8\ m/s}{7000*10^9\ m^2}$$

$$\frac{1}{\rho} = 0.333\,\frac{8.3}{7000}\ 1/m$$

The Magnetic Guide Field



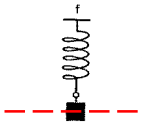
field map of a storage ring dipole magnet

$\rho = 2.53\ km$ → $2\pi\rho = 17.6\ km$
$\approx 66\%$

$B \approx 1 \ldots 8\ T$

rule of thumb: $\frac{1}{\rho} \approx 0.3\,\frac{B[T]}{p[GeV/c]}$ *„normalised bending strength“*

3.) Focusing Forces: Quadrupole Magnets:



$$m * \frac{d^2x}{dt^2} = -c * x$$

classical mechanics: pendulum

required: *focusing forces to keep trajectories in vicinity of the ideal orbit*
linear increasing Lorentz force
linear increasing magnetic field

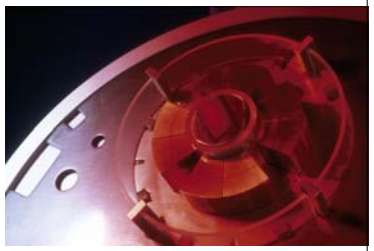
$$B_y = g\,x \qquad B_x = g\,y$$

normalised quadrupole field:

$$\longrightarrow \quad k = \frac{g}{p/e}$$

simple rule:

$$k = 0.3\,\frac{g(T/m)}{p(GeV/c)}$$



LHC main quadrupole magnet

$$g \approx 25 \ldots 220 \;\; T/m$$

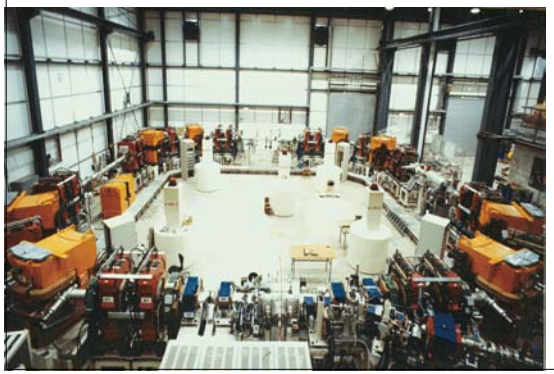
what about the vertical plane:
... Maxwell

$$\vec{\nabla} \times \vec{B} = \cancel{\vec{j}} + \cancel{\frac{\partial \vec{E}}{\partial t}} = 0 \quad \Rightarrow \quad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = g$$

4.) The Equation of Motion:

$$\frac{B(x)}{p/e} = \frac{1}{\rho} + k\,x + \cancel{\frac{1}{2!}m\,x^2} + \cancel{\frac{1}{3!}n\,x^3} + \ldots$$

only terms linear in x, y taken into account *dipole fields*
quadrupole fields



Separate Function Machines:

Split the magnets and optimise them according to their job:

bending, focusing etc

Example:
heavy ion storage ring TSR

* *man sieht nur dipole und quads → linear*

The equation of motion:

Linear approximation:

** ideal particle* → *design orbit*

** any other particle* → *coordinates x, y* *small quantities*
x,y << ρ

→ *magnetic guide field: only linear terms in x & y of B have to be taken into account*

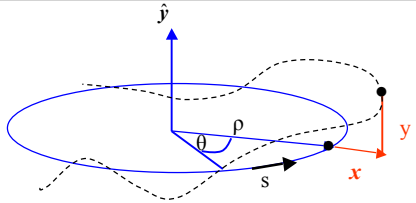
Taylor Expansion of the B field:

$$B_y(x) = B_{y0} + \frac{dB_y}{dx}x + \frac{1}{2!}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{eg''}{dx^3} + \ldots$$

normalise to momentum
p/e = Bρ

$$\frac{B(x)}{p/e} = \frac{B_0}{B_0\rho} + \frac{g*x}{p/e} + \frac{1}{2!}\frac{eg'}{p/e} + \frac{1}{3!}\frac{eg''}{p/e} + \ldots$$

Equation of Motion:



Consider local segment of a particle trajectory
... and remember the old days:
(Goldstein page 27)

radial acceleration:

$$a_r = \frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2$$

Ideal orbit: $\rho = const, \quad \frac{d\rho}{dt} = 0$

Force: $F = m\rho\left(\frac{d\theta}{dt}\right)^2 = m\rho\omega^2$

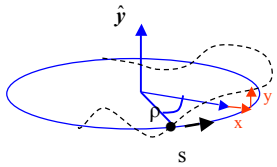
$$F = mv^2/\rho$$

general trajectory: $\rho \rightarrow \rho + x$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

1 2



1 $\frac{d^2}{dt^2}(x+\rho) = \frac{d^2}{dt^2}x$ *… as ρ = const*

2 *remember: x ≈ mm , ρ ≈ m …* → *develop for small x*

$$\frac{1}{x+\rho} \approx \frac{1}{\rho}\left(1-\frac{x}{\rho}\right)$$

Taylor Expansion

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \ldots$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

guide field in linear approx.

$$B_y = B_0 + x\frac{\partial B_y}{\partial x}$$

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = ev\left\{B_0 + x\frac{\partial B_y}{\partial x}\right\} \qquad : m$$

$$\frac{d^2x}{dt^2} - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m}$$

independent variable: t → s

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{ds}\frac{ds}{dt}\right) = \frac{d}{ds}\underbrace{\left(\frac{dx}{ds}\right.}_{x'}\underbrace{\left.\frac{ds}{dt}\right)}_{v}\frac{ds}{dt}$$

$$\frac{d^2x}{dt^2} = x''\,v^2 + \cancel{\frac{dx}{ds}\frac{dv}{ds}v}$$

$$x''v^2 - \frac{v^2}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\,v\,B_0}{m} + \frac{e\,v\,x\,g}{m} \qquad : v^2$$

$$x'' - \frac{1}{\rho}\left(1 - \frac{x}{\rho}\right) = \frac{e\ B_0}{mv} + \frac{e\ x\ g}{mv}$$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x\ g}{p/e}$$

$$x'' - \cancel{\frac{1}{\rho}} + \frac{x}{\rho^2} = -\cancel{\frac{1}{\rho}} + k\ x$$

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = 0$$

$m\ v = p$

normalize to momentum of particle

$$\frac{B_0}{p/e} = -\frac{1}{\rho}$$

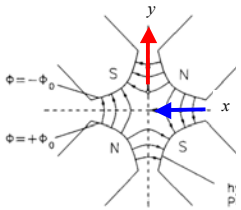
$$\frac{g}{p/e} = k$$

* *Equation for the vertical motion:*

$\frac{1}{\rho^2} = 0$ *no dipoles … in general …*

$k \leftrightarrow -k$ *quadrupole field changes sign*

$$y'' + k\ y = 0$$



5.) Solution of Trajectory Equations

normalise magnet fields to momentum
(remember: $B*\rho = p/q$)

Dipole Magnet

$$\frac{B}{p/q} = \frac{B}{B\rho} = \frac{1}{\rho}$$

Quadrupole Magnet

$$k := \frac{g}{p/q}$$

Solution of Trajectory Equations

Define … hor. plane: K= 1/ρ² + k
… vert. Plane: K = - k

$$x'' + K\,x = 0$$

Differential Equation of harmonic oscillator … with spring constant K

Ansatz: $x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s)$

general solution: linear combination of two independent solutions

$$x'(s) = -a_1\omega \sin(\omega s) + a_2\omega \cos(\omega s)$$

$$x''(s) = -a_1\omega^2 \cos(\omega s) - a_2\omega^2 \sin(\omega s) = -\omega^2 x(s) \longrightarrow \omega = \sqrt{K}$$

general solution:

$$x(s) = a_1 \cos(\sqrt{K}s) + a_2 \sin(\sqrt{K}s)$$

determine a_1, a_2 by boundary conditions:

$$s = 0 \longrightarrow \begin{cases} x(0) = x_0 \quad , \quad a_1 = x_0 \\ x'(0) = x_0' \quad , \quad a_2 = \dfrac{x_0'}{\sqrt{K}} \end{cases}$$

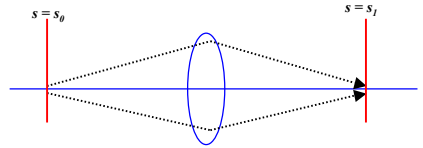
Hor. Focusing Quadrupole K > 0:

$$x(s) = x_0 \cdot \cos(\sqrt{|K|}s) + x_0' \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s)$$

$$x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|}s) + x_0' \cdot \cos(\sqrt{|K|}s)$$

For convenience expressed in matrix formalism:

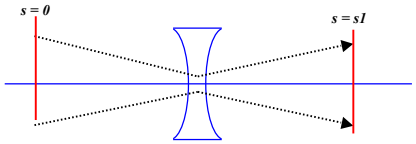
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{foc} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

hor. defocusing quadrupole:

$$x'' - K\,x = 0$$



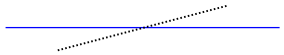
Remember from school:

$$f(s) = \cosh(s) \quad , \quad f'(s) = \sinh(s)$$

$$x(s) = x_0 \cdot \cosh(\sqrt{|K|}s) + x_0' \cdot \sinh(\sqrt{|K|}\,s)$$

$$M_{defoc} = \begin{pmatrix} \cosh\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}l \\ \sqrt{|K|}\sinh\sqrt{|K|}l & \cosh\sqrt{|K|}l \end{pmatrix}$$

drift space:



K = 0

$$x(s) = x_0 + x_0' * s$$

$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

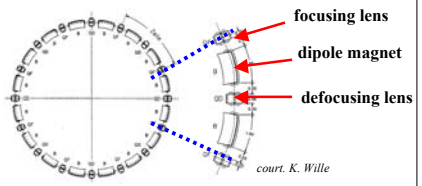
! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & y is uncoupled“

Transformation through a system of lattice elements

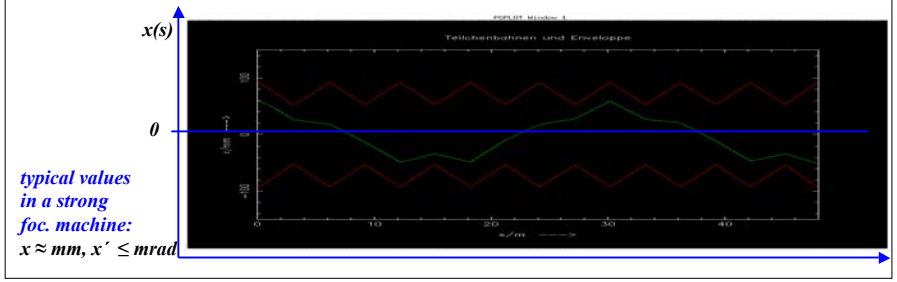
combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

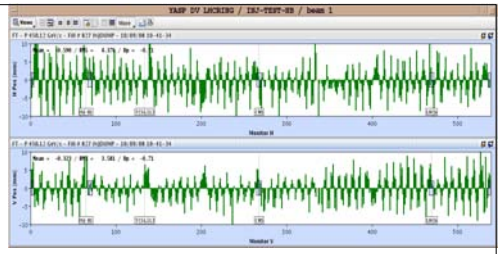
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



in each accelerator element the particle trajectory corresponds to the movement of a harmonic oscillator „



6.) Orbit & Tune:

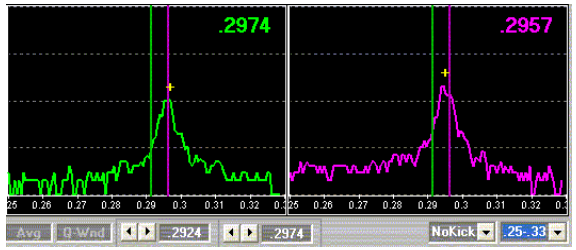


Tune: number of oscillations per turn

64.31
59.32

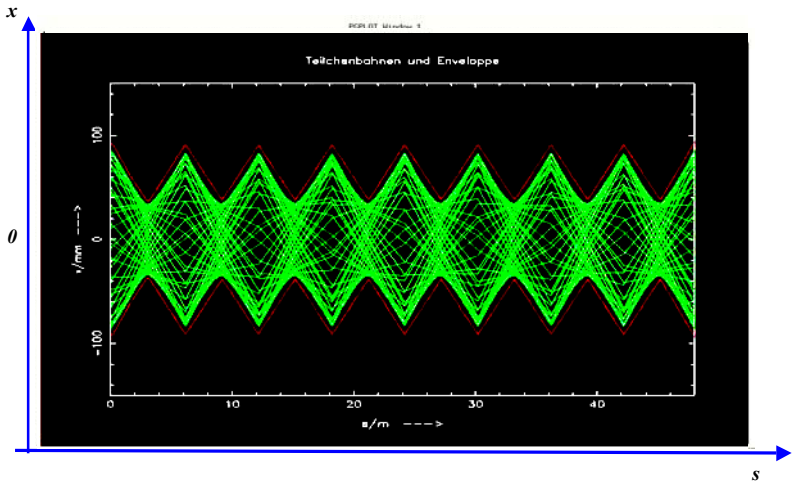
Relevant for beam stability:
non integer part

LHC revolution frequency: 11.3 kHz 0.31*11.3 = 3.5*kHz*



Question: what will happen, if the particle performs a second turn ?

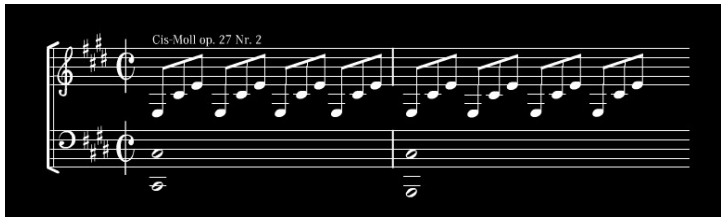
... or a third one or ... 10^{10} turns



19th century:

Ludwig van Beethoven: „Mondschein Sonate“

Sonate Nr. 14 in cis-Moll (op. 27/II, 1801)



Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill‘s equation“

Example: particle motion with periodic coefficient

equation of motion: $x''(s) - k(s)x(s) = 0$

restoring force $\neq$ const,
k(s) = depending on the position s
k(s+L) = k(s), periodic function

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position s in the ring.

7.) The Beta Function

General solution of Hill´s equation:

(i) $$x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\cdot\cos(\psi(s)+\phi)$$

ε, Φ = integration constants determined by initial conditions

β(s) periodic function given by focusing properties of the lattice ↔ quadrupoles

$$\beta(s+L) = \beta(s)$$

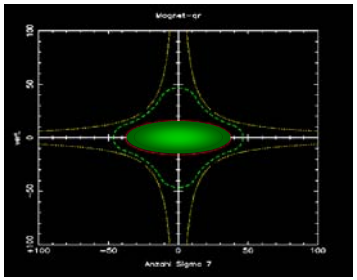
Inserting (i) into the equation of motion …

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Ψ(s) = „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.
For one complete revolution: number of oscillations per turn „Tune“

$$Q_y = \frac{1}{2\pi}\oint\frac{ds}{\beta(s)}$$

The Beta Function

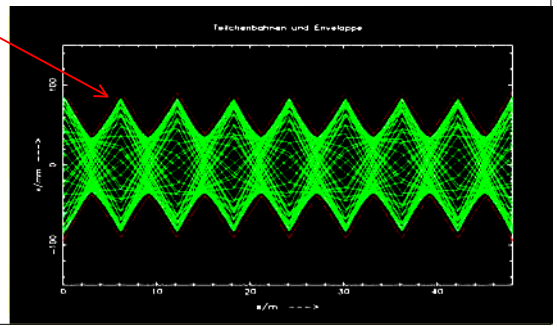


Amplitude of a particle trajectory:

$$x(s) = \sqrt{\varepsilon} * \sqrt{\beta(s)} * \cos(\psi(s)+\varphi)$$

Maximum size of a particle amplitude

$$\hat{x}(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}$$



β determines the beam size
(… the envelope of all particle trajectories at a given position "s" in the storage ring.

It reflects the periodicity of the magnet structure.

8.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

$$\begin{cases} (1) & x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\cos(\psi(s)+\phi) \\ (2) & x'(s) = -\dfrac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left\{\alpha(s)\cos(\psi(s)+\phi)+\sin(\psi(s)+\phi)\right\} \end{cases}$$

from (1) we get

$$\cos(\psi(s)+\phi) = \frac{x(s)}{\sqrt{\varepsilon}\,\sqrt{\beta(s)}}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

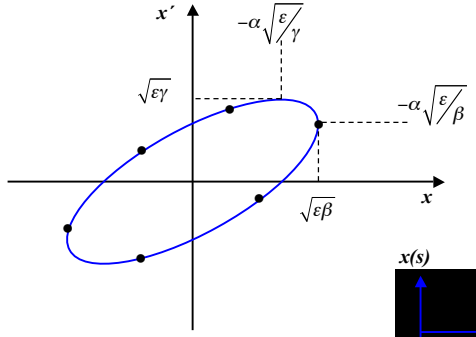
Insert into (2) and solve for ε

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$

**** ε is a constant of the motion … it is independent of „s“***
**** parametric representation of an ellipse in the x x‘ space***
**** shape and orientation of ellipse are given by α, β, γ***

Beam Emittance and Phase Space Ellipse

$$\varepsilon = \gamma(s)*x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



***Liouville:** in reasonable storage rings area in phase space is constant.*

A = π*ε=const

ε beam emittance = woozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.
Scientifiquely spoken: area covered in transverse x, x´ phase space … and it is constant !!!

Phase Space Ellipse

particel trajectory: $x(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\cos\{\psi(s)+\phi\}$

max. Amplitude: $\hat{x}(s) = \sqrt{\varepsilon\beta}$ → *x´ at that position …?*

… put $\hat{x}(s)$ *into* $\boldsymbol{\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)}$ *and solve for x´*

$$\varepsilon = \gamma\cdot\varepsilon\beta + 2\alpha\sqrt{\varepsilon\beta}\cdot x' + \beta x'^2$$

→ $x' = -\alpha\cdot\sqrt{\varepsilon/\beta}$

✻ *A high β-function means a large beam size and a small beam divergence.* !
… et vice versa !!!

✻ *In the middle of a quadrupole β = maximum,*
α = zero } $x' = 0$
… and the ellipse is flat

Phase Space Ellipse

$$\boldsymbol{\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)}$$

$$\alpha(s) = \frac{-1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

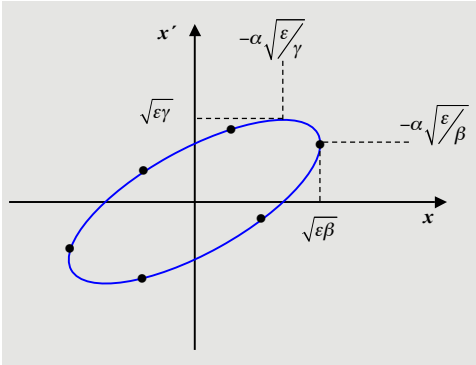
→ $\varepsilon = \dfrac{x^2}{\beta} + \dfrac{\alpha^2 x^2}{\beta} + 2\alpha\cdot xx' + \beta\cdot x'^2$

… solve for x´ $x'_{1,2} = \dfrac{-\alpha\cdot x \pm\sqrt{\varepsilon\beta - x^2}}{\beta}$

… and determine $\hat{x}'$ *via:* $\dfrac{dx'}{dx} = 0$

→ $\hat{x}' = \sqrt{\varepsilon\gamma}$

→ $\hat{x} = \pm\alpha\sqrt{\varepsilon/\gamma}$

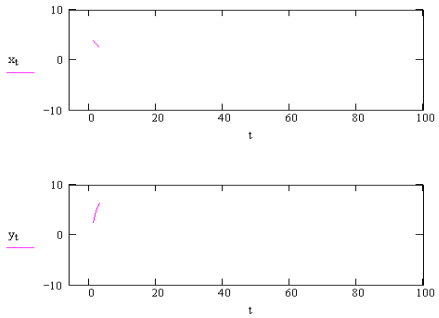


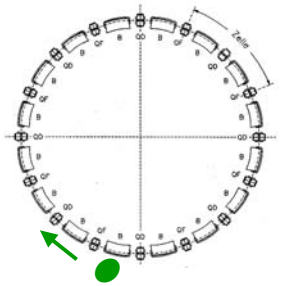
shape and orientation of the phase space ellipse depend on the Twiss parameters β α γ

Particle Tracking in a Storage Ring

Calculate x, x´ for each linear accelerator element according to matrix formalism

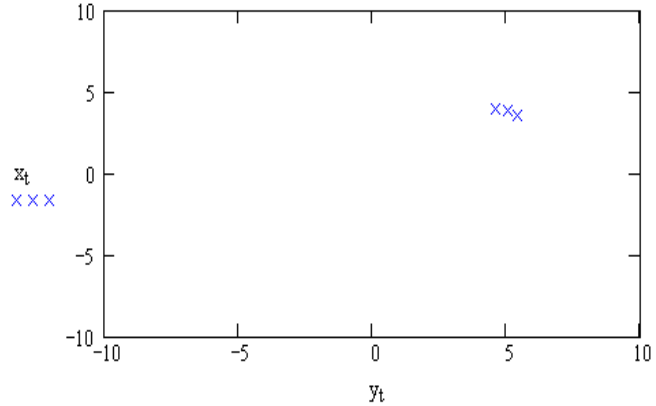
plot x, x´as a function of „s"



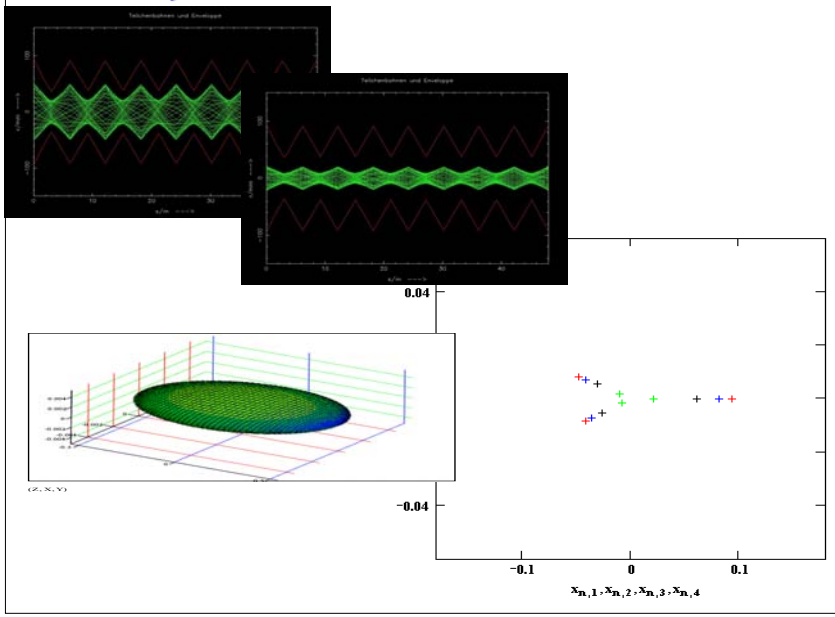


… and now the ellipse:

note for each turn x, x´at a given position „s_1" and plot in the phase space diagram

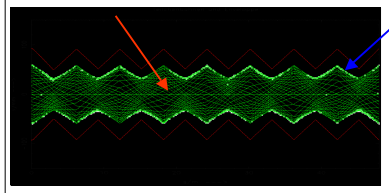


Emittance of the Particle Ensemble:



Emittance of the Particle Ensemble:

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \qquad \hat{x}(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}$$



single particle trajectories, N ≈ 10 ^{11} per bunch

Gauß Particle Distribution:

$$\rho(x) = \frac{N \cdot e}{\sqrt{2\pi}\sigma_x} \cdot e^{-\frac{1}{2}\frac{x^2}{\sigma_x^2}}$$

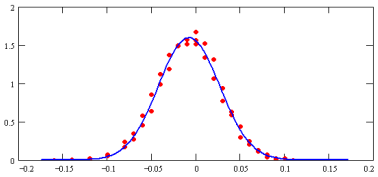
particle at distance 1 σ from centre
↔ 68.3 % of all beam particles

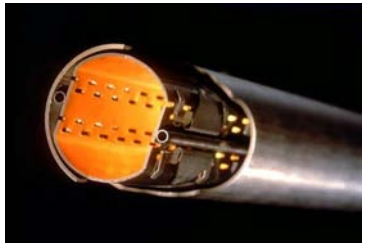
LHC:

$$\beta = 180\,m$$

$$\varepsilon = 5 * 10^{-10}\,m\,rad$$

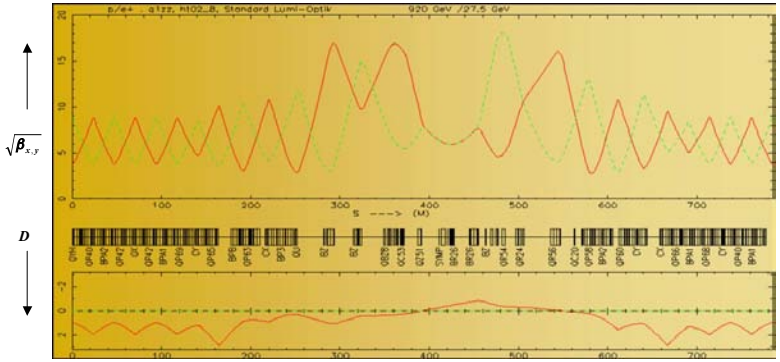
$$\sigma = \sqrt{\varepsilon * \beta} = \sqrt{5 * 10^{-10}\,m * 180\,m} = 0.3\,mm$$





aperture requirements: $r_0 = 12 * \sigma$

III.) Lattice Design in Particle Accelerators



1952: Courant, Livingston, Snyder:

Theory of strong focusing in particle beams

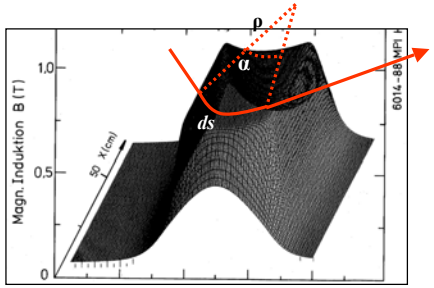
9.) Lattice Design:

„… how to build a storage ring“

$$B\,\rho = p/q$$

Circular Orbit: dipole magnets to define the geometry

$$\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho}$$



field map of a storage ring dipole magnet

The angle run out in one revolution must be 2π, so

… for a full circle $\alpha = \frac{\int Bdl}{B\,\rho} = 2\pi \quad \rightarrow \quad \int Bdl = 2\pi\,\frac{p}{q}$ *… defines the integrated dipole field around the machine.*

Nota bene: $\frac{\Delta B}{B} \approx 10^{-4}$ *is usually required !!*

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\,p/e$$

$$B \approx \frac{2\pi\;\;7000\;\;10^9\,eV}{1232\;\;15\,m\;\;3\;10^8\,\frac{m}{s}\;\;e} = 8.3\,Tesla$$

Recapitulation: storage ring elements
... the story with the matrices !!!

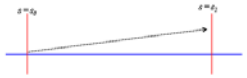
Equation of Motion:

$$x'' + K\,x = 0 \qquad K = 1/\rho^2 - k \;\ldots\; \textit{hor. plane:}$$

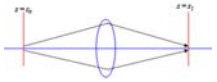
$$K = k \qquad \ldots\; \textit{vert. Plane:}$$

Solution of Trajectory Equations

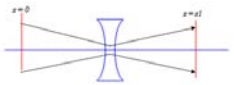
$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$



$$M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$



$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l) & \cos(\sqrt{|K|}l) \end{pmatrix}$$



$$M_{defoc} = \begin{pmatrix} \cosh(\sqrt{|K|}l) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}l) \\ \sqrt{|K|}\sinh(\sqrt{|K|}l) & \cosh(\sqrt{|K|}l) \end{pmatrix}$$

$$M_{total} = M_{QF} * M_D * M_B * M_D * M_{QD} * M_D * \;\ldots$$

10.) Transfer Matrix M

… yes we had the topic already

general solution of Hill´s equation

$$\begin{cases} x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cos\{\psi(s)+\phi\} \\ x'(s) = \dfrac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}}\left[\alpha(s)\cos\{\psi(s)+\phi\}+\sin\{\psi(s)+\phi\}\right] \end{cases}$$

remember the trigonometrical gymnastics: sin(a + b) = … etc

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta_s}\left(\cos\psi_s\cos\phi - \sin\psi_s\sin\phi\right)$$

$$x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}}\left[\alpha_s\cos\psi_s\cos\phi - \alpha_s\sin\psi_s\sin\phi + \sin\psi_s\cos\phi + \cos\psi_s\sin\phi\right]$$

starting at point $s(0) = s_0$, where we put $\Psi(0) = 0$

$$\left.\begin{aligned} \cos\phi &= \frac{x_0}{\sqrt{\varepsilon\beta_0}}\quad, \\ \sin\phi &= -\frac{1}{\sqrt{\varepsilon}}\left(x_0'\sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}}\right) \end{aligned}\right\}$$

inserting above …

$$x(s) = \sqrt{\frac{\beta_s}{\beta_0}}\left\{\cos\psi_s + \alpha_0\sin\psi_s\right\}x_0 + \left\{\sqrt{\beta_s\beta_0}\sin\psi_s\right\}x_0'$$

$$x'(s) = \frac{1}{\sqrt{\beta_s\beta_0}}\left\{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s\right\}x_0 + \sqrt{\frac{\beta_0}{\beta_s}}\left\{\cos\psi_s - \alpha_s\sin\psi_s\right\}x_0'$$

which can be expressed ... for convenience ... in matrix form

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M\begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$M = \begin{pmatrix} \sqrt{\dfrac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0\sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \dfrac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

** we can calculate the single particle trajectories between two locations in the ring, if we know the α β γ at these positions.*

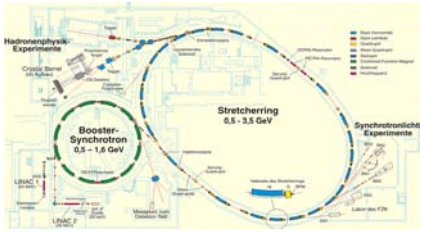
** and nothing but the α β γ at these positions.*

* … !

* *Äquivalenz der Matrizen*

11.) Periodic Lattices

$$M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s \beta_0}\,\sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta s}}\left(\cos\psi_s - \alpha_s \sin\psi_s\right) \end{pmatrix}$$



ELSA Electron Storage Ring

„This rather formidable looking matrix simplifies considerably if we consider one complete revolution …“

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_{turn} & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

$$\psi_{turn} = \int_s^{s+L} \frac{ds}{\beta(s)}$$

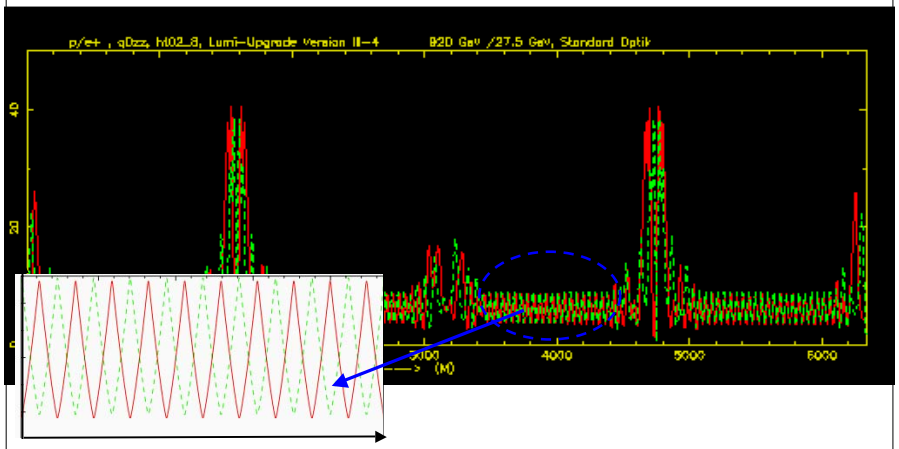
ψ_{turn} = phase advance per period

Tune: Phase advance per turn in units of 2π

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

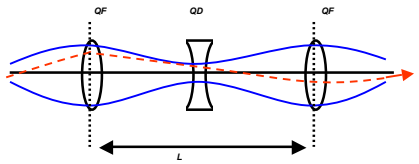
FoDo-Lattice

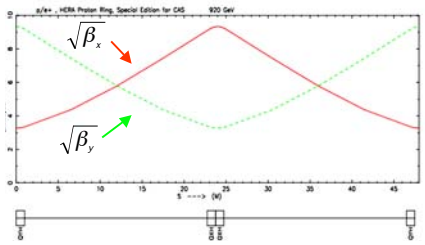
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in .
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)



Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell μ = 45°,
→ calculate the twiss parameters for a periodic solution

Periodic solution of a FoDo Cell





Output of the optics program:

Nr	*Type*	*Length*	*Strength*	β_x	α_x	ψ_x	β_y	α_y	ψ_y
		m	*1/m2*	*m*		*1/2π*	*m*		*1/2π*
0	*IP*	*0,000*	*0,000*	*11,611*	*0,000*	*0,000*	*5,295*	*0,000*	*0,000*
1	*QFH*	*0,250*	*-0,541*	*11,228*	*1,514*	*0,004*	*5,488*	*-0,781*	*0,007*
2	*QD*	*3,251*	*0,541*	*5,488*	*-0,781*	*0,070*	*11,228*	*1,514*	*0,066*
3	*QFH*	*6,002*	*-0,541*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*
4	*IP*	*6,002*	*0,000*	*11,611*	*0,000*	*0,125*	*5,295*	*0,000*	*0,125*

Q_X= *0,125* Q_Y= *0,125* → $0.125 * 2\pi = 45°$

Can we understand, what the optics code is doing?

matrices $M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}l_q) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}l_q) \\ -\sqrt{|K|}\sin(\sqrt{|K|}l_q) & \cos(\sqrt{|K|}l_q) \end{pmatrix}$ $\quad M_{drift} = \begin{pmatrix} 1 & l_d \\ 0 & 1 \end{pmatrix}$

strength and length of the FoDo elements

K = +/- 0.54102 m^{-2}
lq = 0.5 m
ld = 2.5 m

The matrix for the complete cell is obtained by multiplication of the element matrices

$$M_{FoDo} = M_{qfh} * M_{ld} * M_{qd} * M_{ld} * M_{qfh}$$

Putting the numbers in and multiplying out ...

$$M_{FoDo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix}$$

The transfer matrix for one period gives us all the information that we need !

Phase advance per cell

$$M(s) = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\gamma\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix} \rightarrow$$

$$\cos(\psi) = \frac{1}{2}\mathit{Trace}(M) = 0.707$$

$$\psi = \mathit{arc}\cos(\frac{1}{2}\mathit{Trace}(M)) = 45°$$

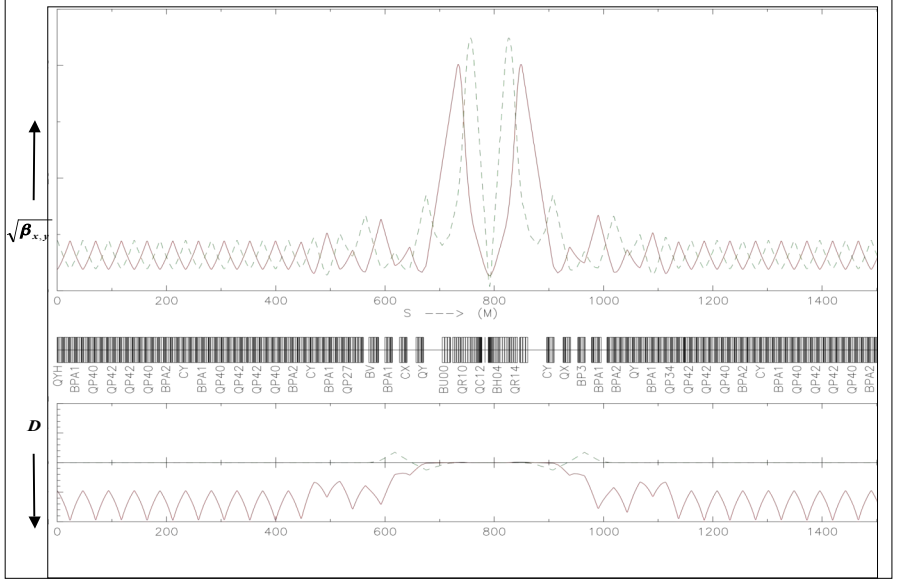
hor β-function

hor α-function

$$\beta = \frac{M_{1,2}}{\sin\psi} = 11.611\,m$$

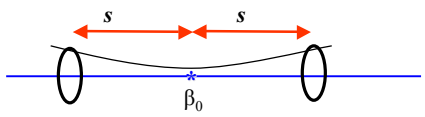
$$\alpha = \frac{M_{1,1} - \cos\psi}{\sin\psi} = 0$$

12.) Insertions



β-Function in a Drift:

let's assume we are at a symmetry point in the center of a drift.

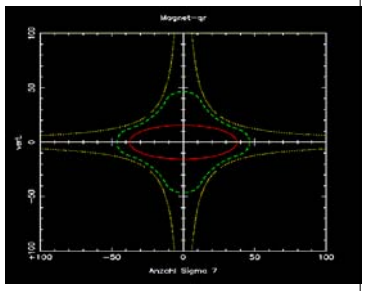


β function in the neighborhood of the symmetry point

$$\beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

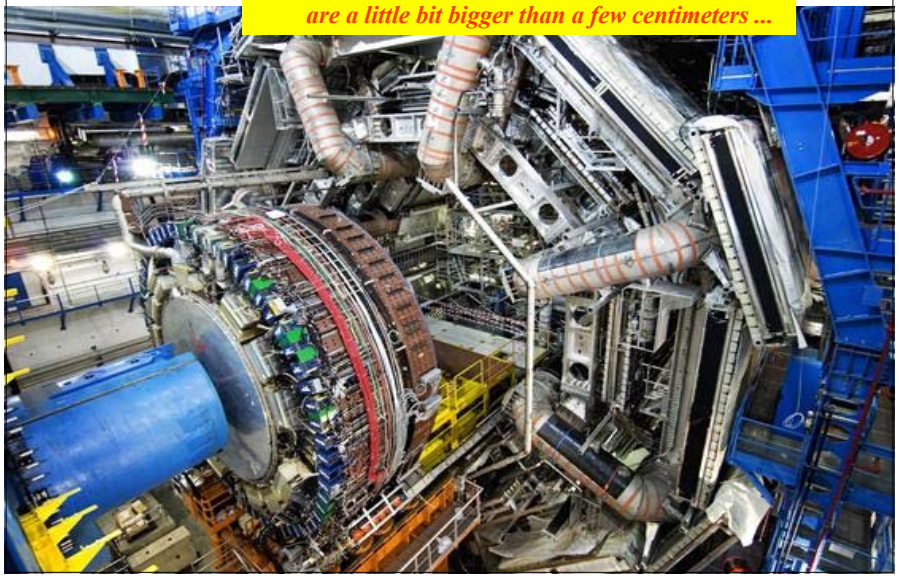
-> keep l as small as possible



7 sigma beam size inside a mini beta quadrupole

... clearly there is an

... unfortunately ... in general high energy detectors that are installed in that drift spaces are a little bit bigger than a few centimeters ...

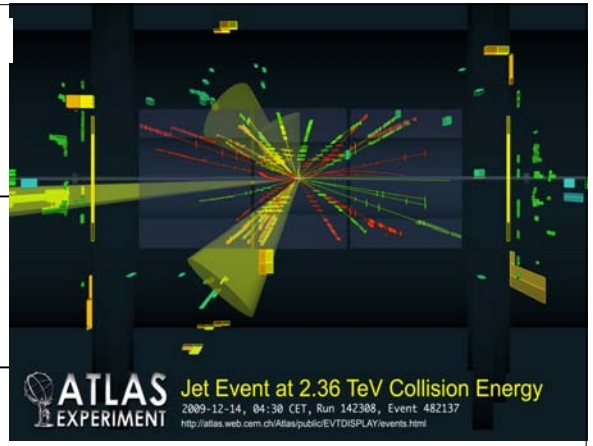


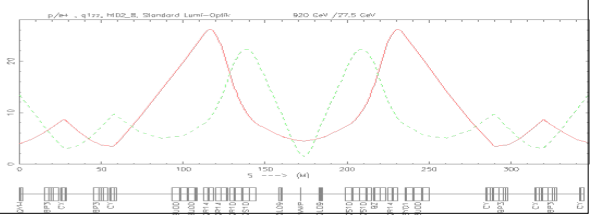
The Mini-β Insertion:

$$R = L * \Sigma_{react}$$

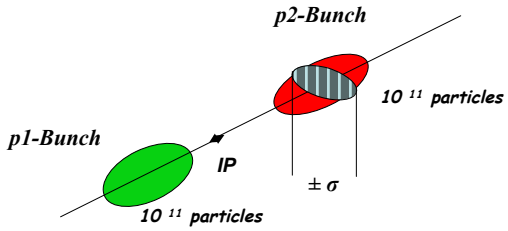
production rate of events
is determined by the
cross section Σ_{react}
and a parameter L that is given
by the design of the accelerator:
… the luminosity

$$L = \frac{1}{4\pi e^2 f_0 \, b} * \frac{I_1 * I_2}{\sigma^*_x * \sigma^*_y}$$





13.) Luminosity



Example: Luminosity run at LHC

$\beta_{x,y} = 0.55\ m$ $\qquad f_0 = 11.245\ kHz$

$\varepsilon_{x,y} = 5 * 10^{-10}\ rad\ m$ $\qquad n_b = 2808$

$\sigma_{x,y} = 17\ \mu m$

$I_p = 584\ mA$

$$L = \frac{1}{4\pi e^2 f_0 n_b} * \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y}$$

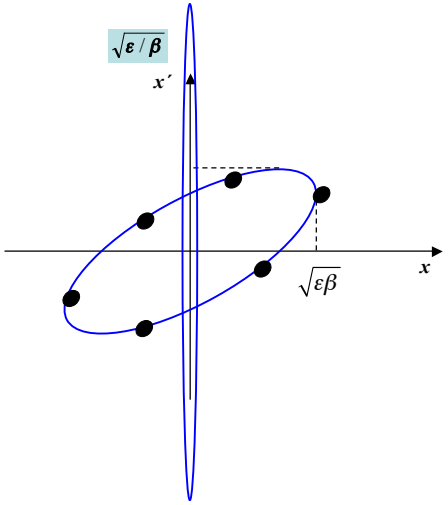
$$L = 1.0 * 10^{34}\ \frac{1}{cm^2 s}$$

beam sizes in the order of my cat's hair !!

Mini-β Insertions: Betafunctions

A mini-β insertion is always a kind of special symmetric drift space.
→greetings from Liouville

the smaller the beam size
the larger the bam divergence



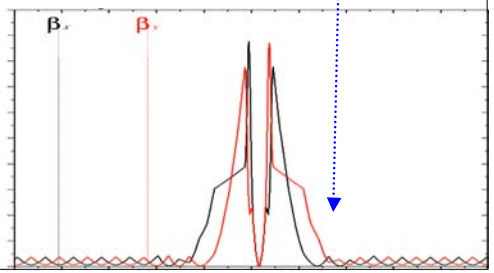
Mini-β Insertions: some guide lines

* calculate the periodic solution in the arc
* introduce the drift space needed for the insertion device (detector ...)
* put a quadrupole doublet (triplet ?) as close as possible
* introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure

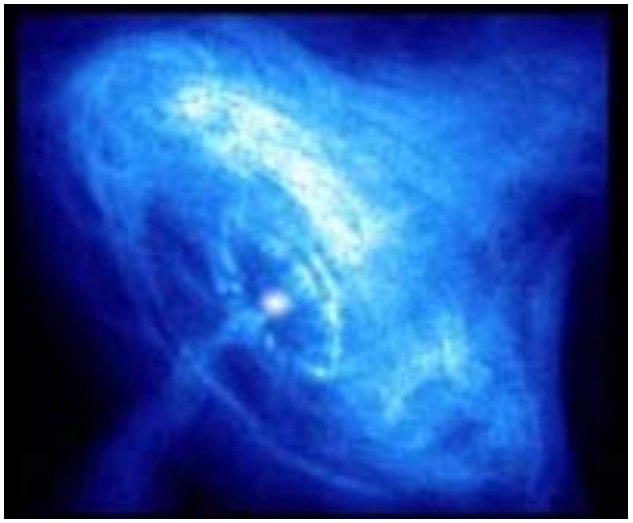
parameters to be optimised & matched to the periodic solution: $\alpha_x,\ \beta_x$ $D_x,\ D_x'$ $\alpha_y,\ \beta_y$ $Q_x,\ Q_y$

8 individually powered quad magnets are needed to match the insertion (... at least)

compact structure of a mini beta triplet



IV) … let´s talk about acceleration

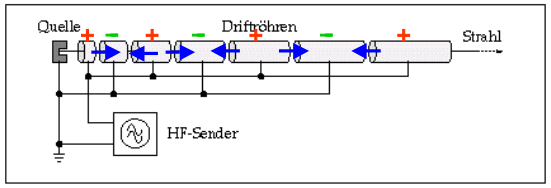


crab nebula,

burst of charged particles $E = 10^{20}$ eV

14.) RF Acceleration

1928, Wideroe



Energy Gain per „Gap“:

$$W = q\,U_0 \sin \omega_{RF} t$$

drift tube structure at a proton linac (GSI Unilac)

500 MHz cavities in an electron storage ring

* *RF Acceleration: multiple application of the same acceleration voltage; brillant idea to gain higher energies*

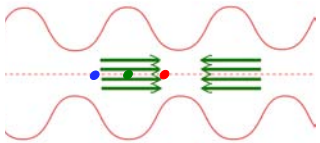
15.) The Acceleration for Δp/p≠0

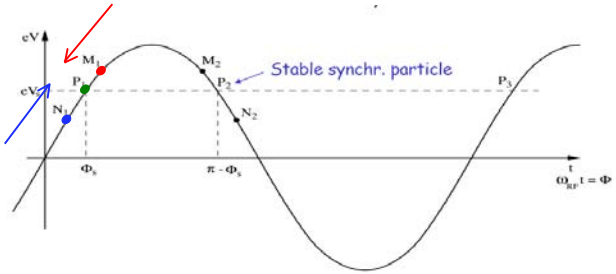
"Phase Focusing" below transition

ideal particle •

particle with Δp/p > 0 • *faster*

particle with Δp/p < 0 • *slower*



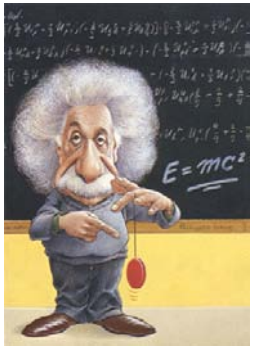


Focussing effect in the longitudinal direction keeping the particles close together ... forming a "bunch"

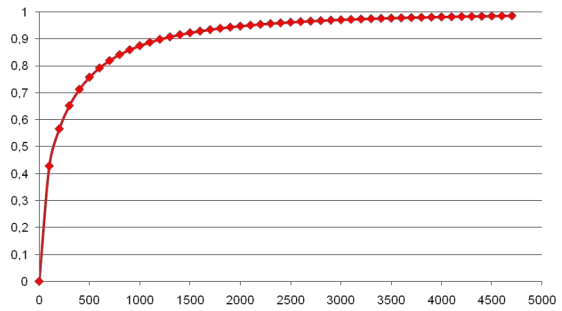
oscillation frequency: $f_s = f_{rev}\sqrt{-\frac{h\alpha_s}{2\pi} * \frac{qU_0 \cos\phi_s}{E_s}}$ *≈ some Hz*

... so sorry, here we need help from Albert:

$$\gamma = \frac{E_{total}}{mc^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \longrightarrow \frac{v}{c} = \sqrt{1-\frac{mc^2}{E^2}}$$



v/c



kinetic energy of a proton

... some when the particles do not get faster anymore

.... but heavier !

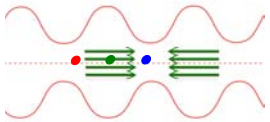
16.) The Acceleration for $\Delta p/p \neq 0$

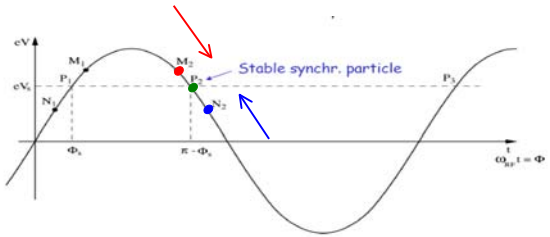
"Phase Focusing" above transition

ideal particle •

particle with $\Delta p/p > 0$ • *heavier*

particle with $\Delta p/p < 0$ • *lighter*





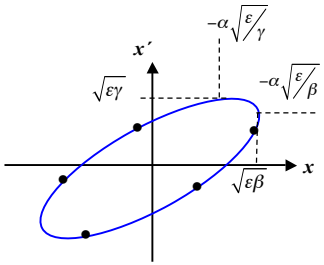
Focussing effect in the longitudinal direction
keeping the particles close together ... forming a "bunch"

... and how do we accelerate now ???

with the dipole magnets !

17.) Liouville during Acceleration

$$\varepsilon = \gamma(s)\, x^2(s) + 2\alpha(s) x(s) x'(s) + \beta(s)\, x'^2(s)$$



Beam Emittance ***corresponds to the area covered in the x, x´ Phase Space Ellipse***

Liouville: ***Area in phase space is constant.***

But so sorry ... $\varepsilon \neq const$!

Classical Mechanics:

phase space ***= diagram of the two canonical variables***
position ***&*** ***momentum***
x $\quad$ p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

q = position = x
p = momentum = γmv = mcγβ$_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\, dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} \qquad \text{where } \beta_x = v_x / c$$

$$\int p dq = mc \int \gamma \beta_x dx$$

$$\int p dq = mc\gamma\beta \underbrace{\int x' dx}_{\varepsilon}$$

$$\Rightarrow \quad \varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

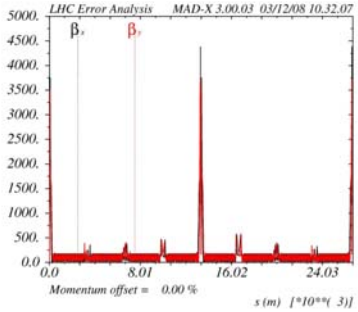
Nota bene:

1.) A proton machine … or an electron linac … needs the highest aperture at injection energy !!!
as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

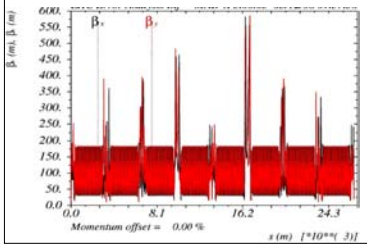
$$\sigma = \sqrt{\varepsilon\beta}$$

2.) At lowest energy the machine will have the major aperture problems,
→ here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
A Mini Beta concept will only be adequate at flat top.



LHC mini beta optics at 7000 GeV

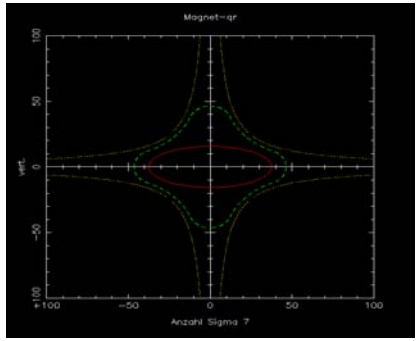


LHC injection optics at 450 GeV

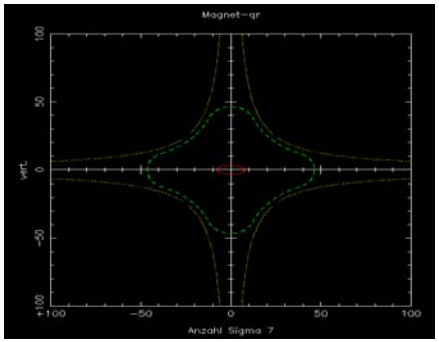
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε (40GeV) $= 1.2 * 10^{-7}$
ε (920GeV) $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

The „ not so ideal world "

18.) The „ $\Delta p / p \neq 0$" Problem

ideal accelerator: *all particles will see the same accelerating voltage.*

→ $\Delta p / p = 0$

„nearly ideal" accelerator: *Cockroft Walton or van de Graaf*

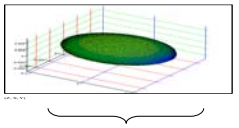
$\Delta p / p \approx 10^{-5}$

Vivitron, Straßbourg, inner structure of the acc. section

MP Tandem van de Graaf Accelerator at MPI for Nucl. Phys. Heidelberg

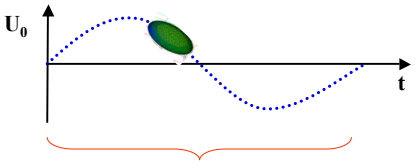
RF Acceleration-Problem: panta rhei !!!

(Heraklit: 540-480 v. Chr.)



Bunch length of Electrons ≈ 1cm

just a stupid (and nearly wrong) example)



$\lambda = 75\ cm$

$$\left.\begin{aligned} \nu &= 400\,MHz \\ c &= \lambda\,\nu \end{aligned}\right\} \lambda = 75\ cm$$

$\sin(90^o) = 1$

$\sin(84^o) = 0.994$

$$\frac{\Delta U}{U} = 6.0\ 10^{-3}$$

typical momentum spread of an electron bunch: $\frac{\Delta p}{p} \approx 1.0\ 10^{-3}$

V.) Errors in Field and Gradient

Dispersive and Chromatic Effects
$\Delta p/p \neq 0$

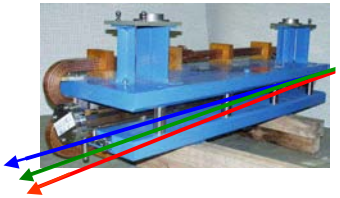
Are there any Problems ???
Sure there are !!!

font colors due to pedagogical reasons

Dispersion and Chromaticity: Magnet Errors for $\Delta p/p \neq 0$

Influence of external fields on the beam: prop. to magn. field & prop. zu 1/p

dipole magnet $\alpha = \frac{\int B\, dl}{p/e}$



$$x_D(s) = D(s)\frac{\Delta p}{p}$$

focusing lens $k = \frac{g}{p/e}$

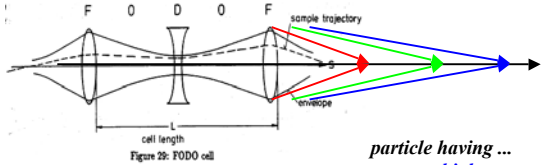
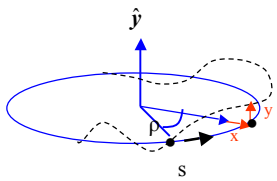


Figure 29: FODO cell

particle having ...
to high energy
to low energy
ideal energy

19.) Dispersion: trajectories for $\Delta p / p \neq 0$

Question: do you remember last session, page 12 ? … sure you do



Force acting on the particle

$$F = m\frac{d^2}{dt^2}(x+\rho) - \frac{mv^2}{x+\rho} = e\,B_y\,v$$

remember: $x \approx mm$, $\rho \approx m$ … → develop for small x

$$m\frac{d^2x}{dt^2} - \frac{mv^2}{\rho}\left(1-\frac{x}{\rho}\right) = eB_y v$$

consider only linear fields, and change independent variable: $t \rightarrow s$ $\qquad B_y = B_0 + x\frac{\partial B_y}{\partial x}$

$$x'' - \frac{1}{\rho}\left(1-\frac{x}{\rho}\right) = \frac{e\,B_0}{mv} + \frac{e\,x\,g}{mv}$$

$p=p_0+\Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error $\qquad \Delta p << p_0 \Rightarrow \frac{1}{p_0+\Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$

$$x'' - \frac{1}{\rho} + \frac{x}{\rho^2} \approx \underbrace{\frac{e\,B_0}{p_0}}_{-\frac{1}{\rho}} - \frac{\Delta p}{p_0^2}eB_0 + \underbrace{\frac{xeg}{p_0}}_{k*x} - \underbrace{xeg\frac{\Delta p}{p_0^2}}_{\approx 0}$$

$$x'' + \frac{x}{\rho^2} \approx \frac{\Delta p}{p_0} * \underbrace{\frac{(-eB_0)}{p_0}}_{\frac{1}{\rho}} + k*x = \frac{\Delta p}{p_0} * \frac{1}{\rho} + k*x$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0}\frac{1}{\rho} \longrightarrow x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0}\frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.
→ inhomogeneous differential equation.

Dispersion:

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \frac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

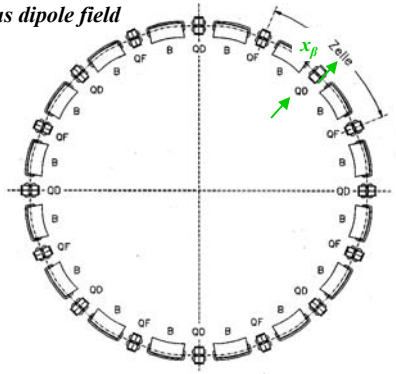
$$D(s) = \frac{x_i(s)}{\Delta p / p}$$

Dispersion function D(s)

* *is that special orbit, an ideal particle would have for Δp/p = 1*
* *the orbit of any particle is the sum of the well known x_β and the dispersion*
* *as D(s) is just another orbit it will be subject to the focusing properties of the lattice*

Dispersion

Example: homogeneous dipole field



it for Δp/p > 0

$$D(s) \cdot \frac{\Delta p}{p}$$

Matrix formalism:

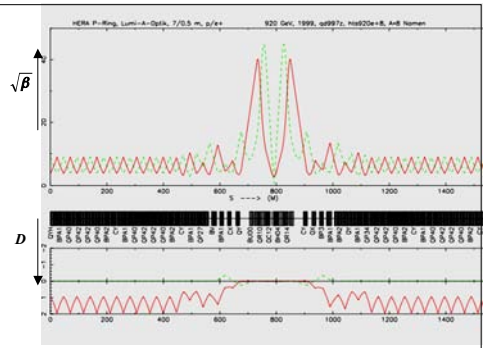
$$x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}$$

$$x(s) = C(s) \cdot x_0 + S(s) \cdot x_0' + D(s) \cdot \frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}_0$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example

$$\left.\begin{aligned} x_\beta &= 1 \ldots 2\ mm \\ D(s) &\approx 1 \ldots 2\ m \\ \Delta p/p &\approx 1 \cdot 10^{-3} \end{aligned}\right\}$$

Amplitude of Orbit oscillation

contribution due to Dispersion ≈ beam size

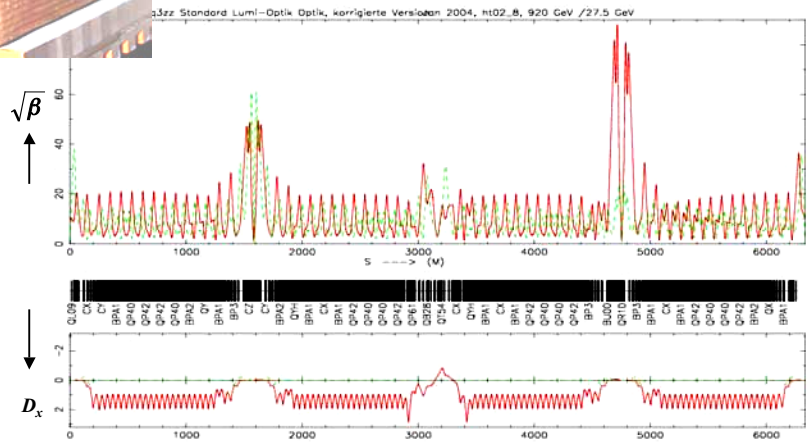
→ Dispersion must vanish at the collision point

!

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})d\tilde{s}$$

20.) Quadrupole Errors



Quadrupole Errors

go back to Lecture I, page 1
single particle trajectory

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M_{QF} * \begin{pmatrix} x \\ x' \end{pmatrix}_1$$

Solution of equation of motion

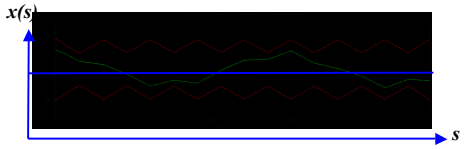
$$x = x_0 \cos(\sqrt{k}\, l_q) + x'_0 \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q)$$

$$M_{QF} = \begin{pmatrix} \cos(\sqrt{k}\, l_q) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}\, l_q) \\ -\sqrt{k} \sin(\sqrt{k}\, l_q) & \cos(\sqrt{k}\, l_q) \end{pmatrix} \quad , \quad M_{thin\,lens} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$M_{turn} = M_{QF} * M_{D1} * M_{QD} * M_{D2} * M_{QF} \cdots$$

Definition: phase advance of the particle oscillation per revolution in units of 2π is called tune

$$Q = \frac{\psi_{turn}}{2\pi}$$



Matrix in Twiss Form

Transfer Matrix from point „0“ in the lattice to point „s“:

$$M(s) = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos\psi_s + \alpha_0 \sin\psi_s) & \sqrt{\beta_s \beta_0} \sin\psi_s \\ \frac{(\alpha_0 - \alpha_s)\cos(\psi_s - (1 + \alpha_0 \alpha_s)\sin\psi_s}{\sqrt{\beta_s \beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}} (\cos(\psi_s - \alpha_0 \sin\psi_s) \end{pmatrix}$$

For one complete turn the Twiss parameters have to obey periodic bundary conditions:

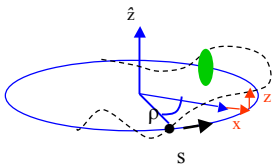
$$\beta(s+L) = \beta(s)$$
$$\alpha(s+L) = \alpha(s)$$
$$\gamma(s+L) = \gamma(s)$$

$$M(s) = \begin{pmatrix} \cos\psi_{turn} + \alpha_s \sin\psi_{turn} & \beta_s \sin\psi_{turn} \\ -\gamma_s \sin\psi_s & \cos\psi_{turn} - \alpha_s \sin\psi_{turn} \end{pmatrix}$$

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

$$M_{dist} = M_{\Delta k} \cdot M_0 = \underbrace{\begin{pmatrix} 1 & 0 \\ \Delta kds & 1 \end{pmatrix}}_{\text{quad error}} \cdot \underbrace{\begin{pmatrix} \cos\psi_{turn} + \alpha \sin\psi_{turn} & \beta \sin\psi_{turn} \\ -\gamma \sin\psi_{turn} & \cos\psi_{turn} - \alpha \sin\psi_{turn} \end{pmatrix}}_{\text{ideal storage ring}}$$



$$M_{dist} = \begin{pmatrix} \cos\psi_0 + \alpha \sin\psi_0 & \beta \sin\psi_0 \\ \Delta kds(\cos\psi_0 + \alpha \sin\psi_0) - \gamma \sin\psi_0 & \Delta kds\, \beta \sin\psi_0 + \cos\psi_0 - \alpha \sin\psi_0 \end{pmatrix}$$

rule for getting the tune

$$Trace(M) = 2\cos\psi = 2\cos\psi_0 + \Delta kds\beta \sin\psi_0$$

Quadrupole error → Tune Shift

$$\psi = \psi_0 + \Delta\psi \longrightarrow \cos(\psi_0 + \Delta\psi) = \cos\psi_0 + \frac{\Delta kds\, \beta \sin\psi_0}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos\psi_0 \underbrace{\cos\Delta\psi}_{\approx 1} - \sin\psi_0 \underbrace{\sin\Delta\psi}_{\approx \Delta\psi} = \cos\psi_0 + \frac{kds\, \beta \sin\psi_0}{2}$$

$$\Delta\psi = \frac{kds\, \beta}{2}$$

and referring to Q instead of ψ:

$$\psi = 2\pi Q$$

$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s)ds}{4\pi}$$

! the tune shift is proportional to the β-function at the quadrupole

!! field quality, power supply tolerances etc are much tighter at places where β is large

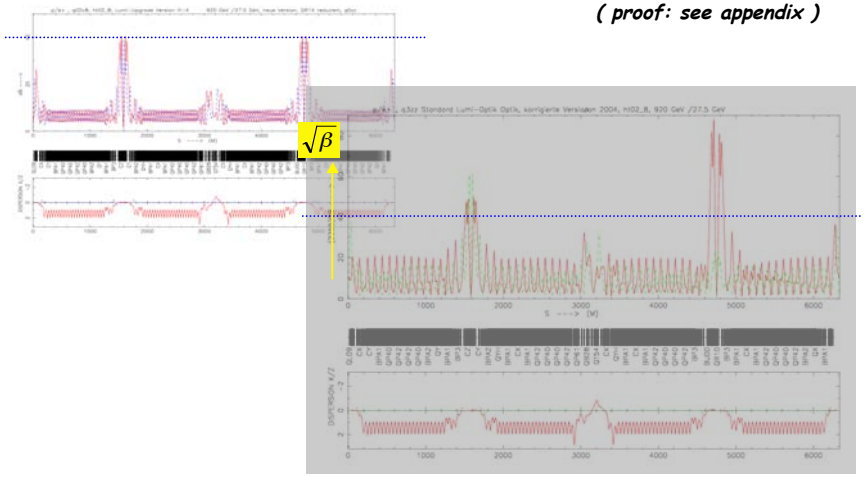
!!! mini beta quads: β ≈ 1900 m
arc quads: β ≈ 80 m

!!!! β is a measure for the sensitivity of the beam

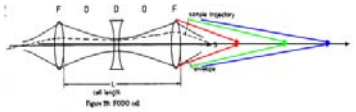
Quadrupole error: Beta Beat

$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta K\cos\left(2|\psi_{s1}-\psi_{s0}|-2\pi Q\right)ds$$

(proof: see appendix)



21.) Chromaticity:
A Quadrupole Error for $\Delta p/p \neq 0$



normalised foc. strength: $k = \frac{g}{p/e}$

in case of a momentum spread: $p = p_0 + \Delta p$

$$k = \frac{eg}{p_0+\Delta p} \approx \frac{e}{p_0}\left(1-\frac{\Delta p}{p_0}\right)g = k_0 + \Delta k \longrightarrow \Delta k = -\frac{\Delta p}{p_0}k_0$$

… which acts like a quadrupole error in the machine and leads to a tune spread:

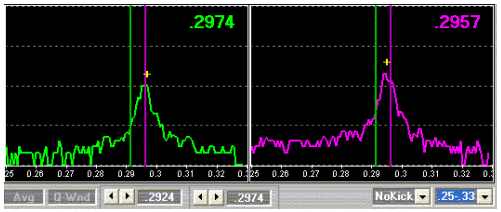
$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0}k_0\beta(s)ds$$

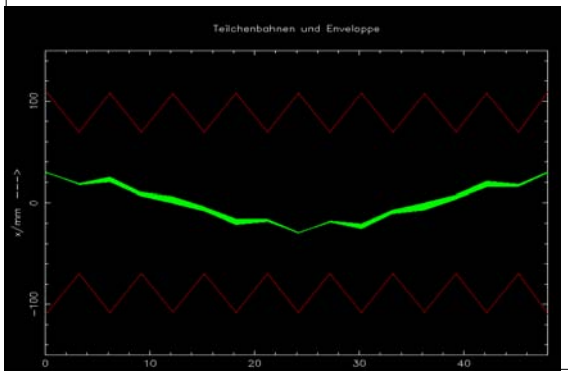
definition of chromaticity:

$$\Delta Q = Q'\frac{\Delta p}{p} \quad ; \quad Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

Where is the Problem ?

Tunes and Resonances





avoid resonance conditions:

$$m\,Q_x + n\,Q_y + l\,Q_s = \text{integer}$$

… for example: 1 Q_x=1

… and now again about Chromaticity:

Problem: chromaticity is generated by the lattice itself !!

Q' is a number indicating the size of the tune spot in the working diagram,
Q' is always created if the beam is focussed
→ it is determined by the focusing strength k of all quadrupoles

$$Q' = -\frac{1}{4\pi}\oint k(s)\beta(s)ds$$

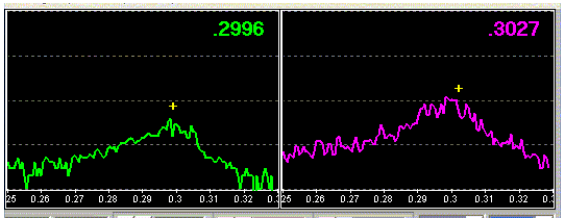
k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

$Q' = 250$
$\Delta p/p = +/- 0.2 * 10^{-3}$
$\Delta Q = 0.256 \ldots 0.36$

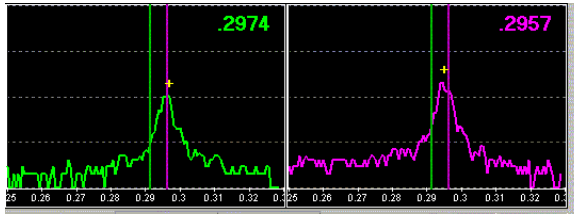
→Some particles get very close to resonances and are lost

in other words: the tune is not a point it is a pancake



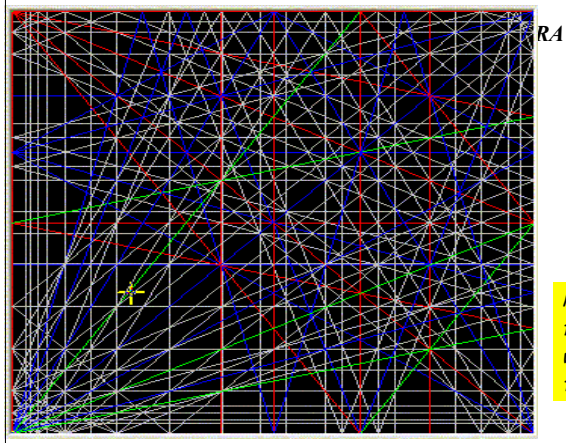
Tune signal for a nearly uncompensated cromaticity (Q' ≈ 20)

Ideal situation: cromaticity well corrected, (Q' ≈ 1)



Tune and Resonances

$$m*Q_x+n*Q_y+l*Q_s = integer$$



RA e Tune diagram up to 3rd order

… and up to 7th order

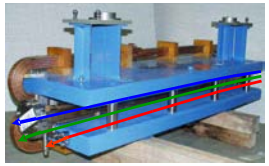
Homework for the operateurs: find a nice place for the tune where against all probability the beam will survive

Correction of Q':

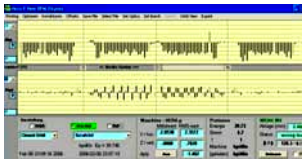
Need: additional quadrupole strength for each momentum deviation Δp/p

1.) sort the particles acording to their momentum

$$x_D(s) = D(s)\frac{\Delta p}{p}$$



… using the dispersion function



2.) apply a magnetic field that rises quadratically with x (sextupole field)

$$\left.\begin{aligned} B_x &= \tilde{g}xz \\ B_z &= \frac{1}{2}\tilde{g}(x^2 - z^2) \end{aligned}\right\} \quad \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x$$

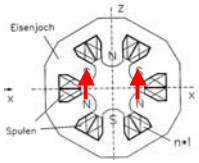
linear rising „gradient“:

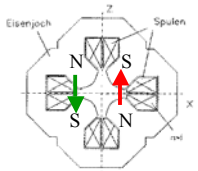
Correction of Q':

k_1 *normalised quadrupole strength*

k_2 *normalised sextupole strength*

Sextupole Magnets:





$$k_1(sext) = \frac{\tilde{g}\,x}{p/e} = k_2 * x$$

$$k_1(sext) = k_2 * D * \frac{\Delta p}{p}$$

corrected chromaticity

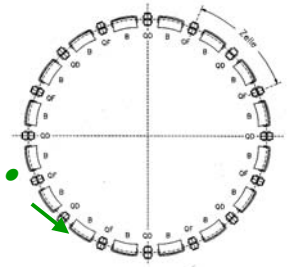
considering a single cell:

$$Q'_{cell\_x} = -\frac{1}{4\pi}\left\{k_{qf}\hat{\beta}_x l_{qf} - k_{qd}\breve{\beta}_x l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

$$Q'_{cell\_y} = -\frac{1}{4\pi}\left\{-k_{qf}\breve{\beta}_y l_{qf} + k_{qd}\hat{\beta}_y l_{qd}\right\} + \frac{1}{4\pi}\sum_{F\,sext} k_2^F l_{sext} D_x^F \beta_x^F - \frac{1}{4\pi}\sum_{D\,sext} k_2^D l_{sext} D_x^D \beta_x^D$$

Clearly there is another problem ...

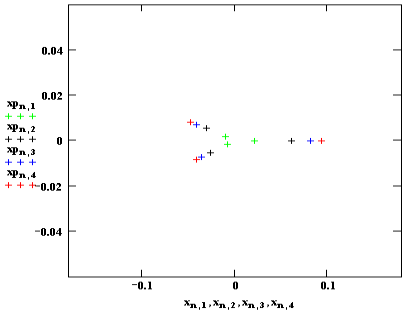
... if it were easy everybody could do it



Again: the phase space ellipse

for each turn write down – at a given position „s" in the ring – the single partilce amplitude x and the angle x´... and plot it.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s1} = M_{turn} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0}$$

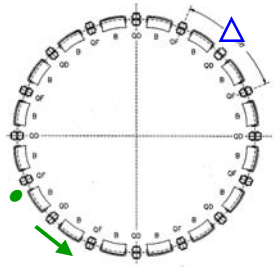


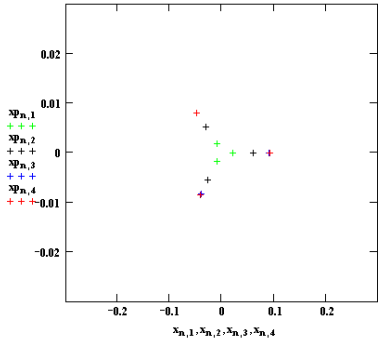
A beam of 4 particles

– each having a slightly different emittance:

Installation of a weak (!!!) sextupole magnet

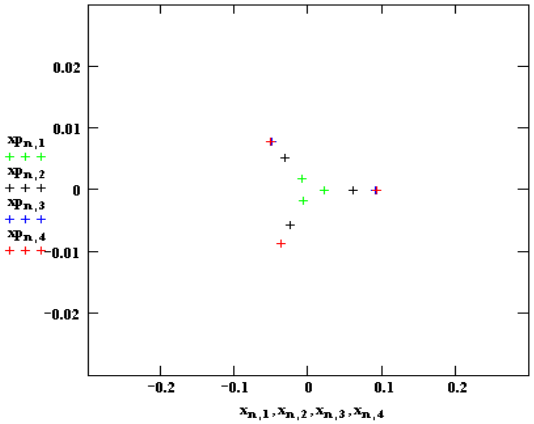
The good news: sextupole fields in accelerators cannot be treated analytically anymore.
→ no equatiuons; instead: Computer simulation
„ particle tracking "



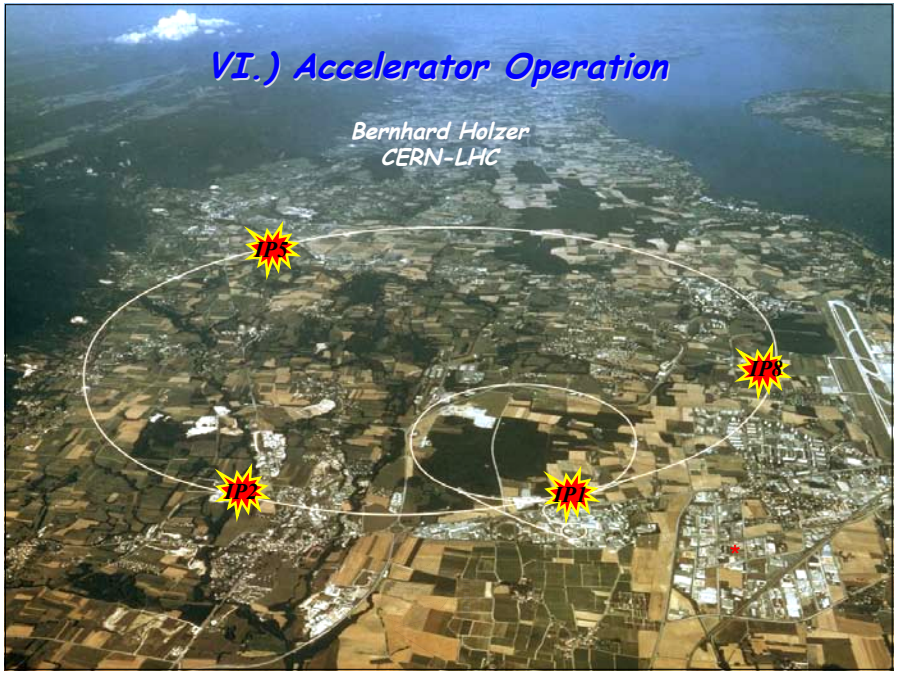


Effect of a strong (!!!) Sextupole …

→ Catastrophy !

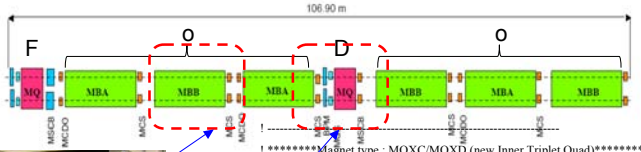


„dynamic aperture“

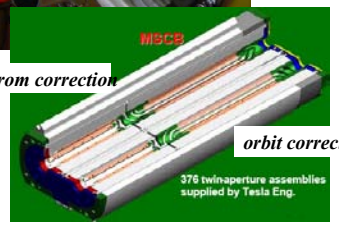


LHC: Basic Layout of the Machine multipole corrector magnets

2, 6, 8, 10, 12 pol
skew & trim quad, chroma 6pol
landau 8 pole



multipole correction



chrom correction

orbit correction

! ----------

! ********Magnet type : MQXC/MQXD (new Inner Triplet Quad)**********

! ----------

bn in collision LHC TDR

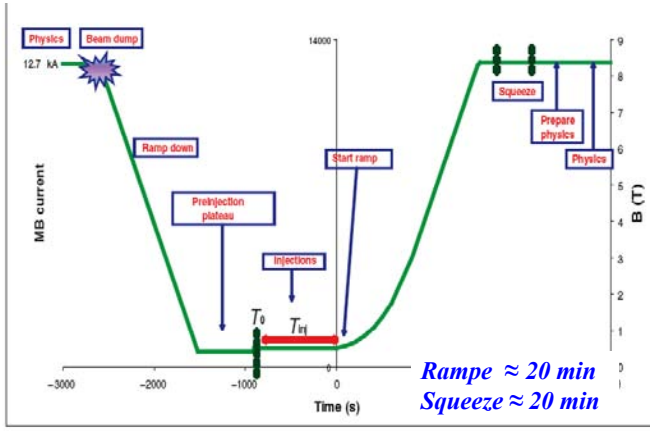
b1M_MQXCD_col := 0.0000 ; b1U_MQXCD_col := 0.0000 ; b1R_MQXCD_col := 0.0000 ;
b2M_MQXCD_col := 0.0000 ; b2U_MQXCD_col := 0.0000 ; b2R_MQXCD_col := 0.0000 ;
b3M_M = 0.8900 ;
b4M_M = 0.6400 ;
b5M_M = 0.4600 ;
b6M_MQXCD_col := 0.0000 ; b6U_MQXCD_col := 1.7700 ; b6R_MQXCD_col := 1.2800 ;
b7M_MQXCD_col := 0.0000 ; b7U_MQXCD_col := 0.2100 ; b7R_MQXCD_col := 0.2100 ;
b8M_MQXCD_col := 0.0000 ; b8U_MQXCD_col := 0.1600 ; b8R_MQXCD_col := 0.1600 ;
b9M_MQXCD_col := 0.0000 ; b9U_MQXCD_col := 0.0800 ;
b10M_MQXCD_col := 0.0000 ; b10U_MQXCD_col := 0.0600 ;
b11M_MQXCD_col := 0.0000 ; b11U_MQXCD_col := 0.0300 ;
b12M_MQXCD_col := 0.0000 ; b12U_MQXCD_col := 0.0200 ; b12R_MQXCD_col := 0.0200 ;
b13M_MQXCD_col := 0.0000 ; b13U_MQXCD_col := 0.0200 ; b13R_MQXCD_col := 0.0100 ;
b14M_MQXCD_col := 0.0000 ; b14U_MQXCD_col := 0.0400 ; b14R_MQXCD_col := 0.0100 ;
b15M_MQXCD_col := 0.0000 ; b15U_MQXCD_col := 0.0000 ; b15R_MQXCD_col := 0.0000 ;

$$x_{co}(s) = \frac{\sqrt{\beta(s)} * \int \frac{1}{\rho_{s1}} \sqrt{\beta_{s1}} * \cos(|\psi_{s1} - \psi_s| - \pi Q)\, ds}{2 \sin \pi Q}$$

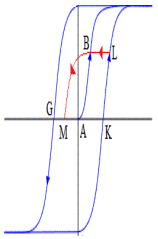
$$Q' = -\frac{1}{4\pi} \oint k(s)\beta(s)\, ds$$

LHC Operation: Magnet Preparation Cycle & Ramp

8 independent sectors**, hysteresis effects, **saturation & remanence
in nc and sc magnets**, synchronisation of the power converters, **magnet model
to describe the transfer functions of every element



Rampe ≈ 20 min
Squeeze ≈ 20 min

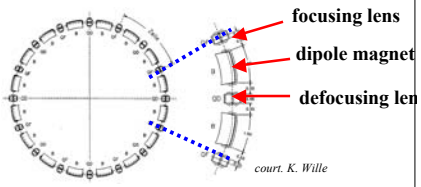


magnet hysteresis

LHC First Turn Steering

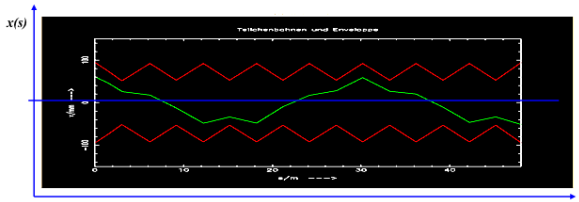
$$M_{total} = M_{QF} * M_D * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s_2, s_1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$



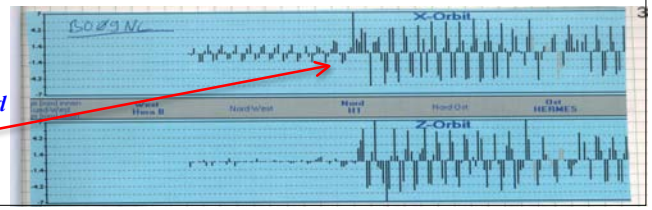
court. K. Wille

in theory
nice harmonic oscillation



in reality:
effect of many localised
orbit distortions

-> correct

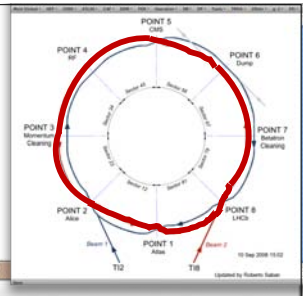


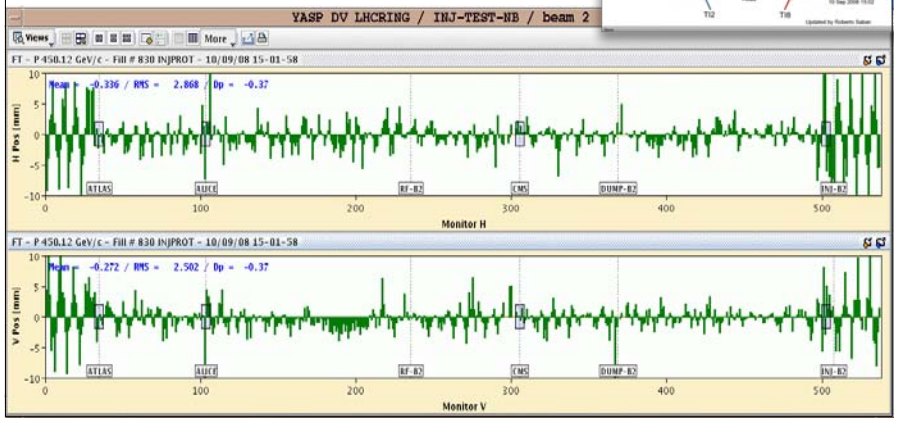
LHC Operation: Beam Commissioning

First turn steering "by sector:"

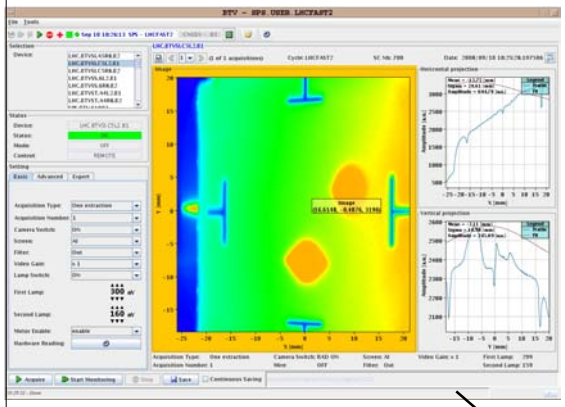
- One beam at the time
- Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.



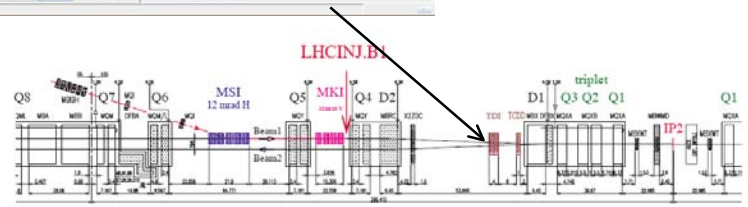


LHC Operation: the First Turn

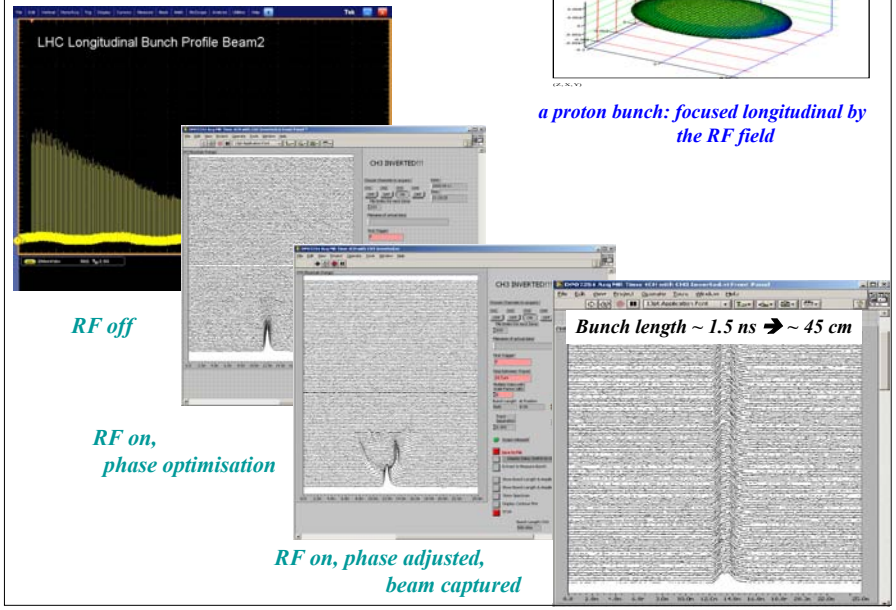


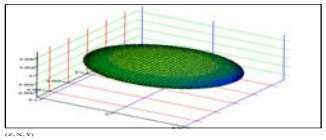
Beam 1 on OTR screen
1st and 2nd turn

Correct x, x',
y, y'
to obtain the Closed Orbit



LHC Commissioning: RF





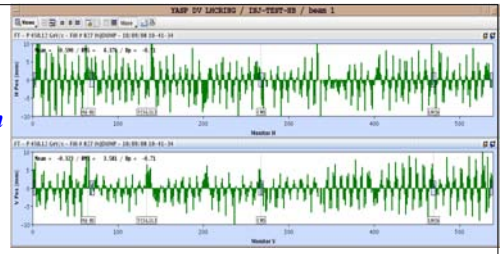
a proton bunch: focused longitudinal by the RF field

RF off

RF on, phase optimisation

RF on, phase adjusted, beam captured

Orbit & Tune:



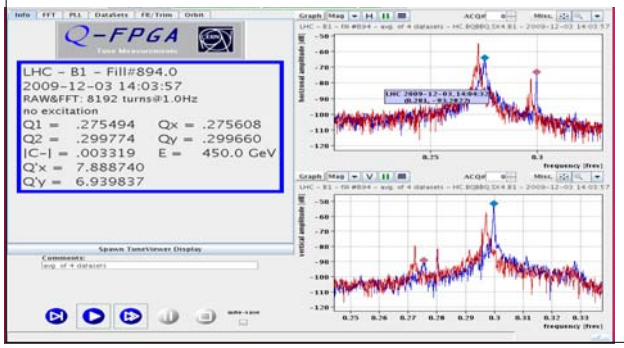
Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

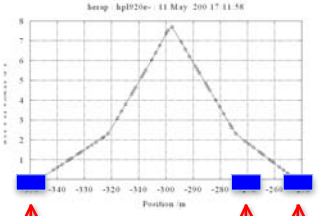
LHC revolution frequency: 11.3 kHz

0.31*11.3 = 3.5*kHz*

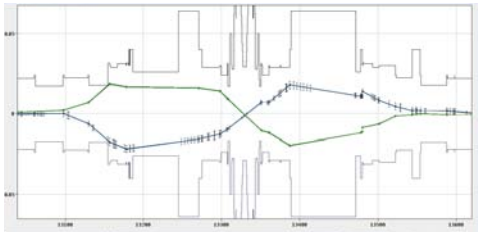


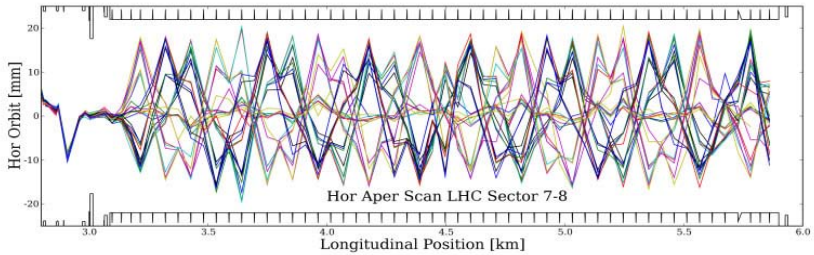
LHC Operation: Aperture Scans

Apply closed orbit bumps until losses indicate the aperture limit
... what about the beam size ?



3 bump to create local orbit distortion



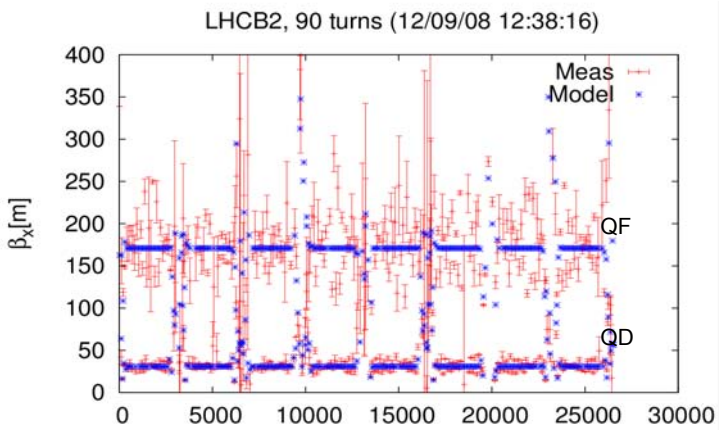


LHC Operation: the First Beam

Measurement of β :

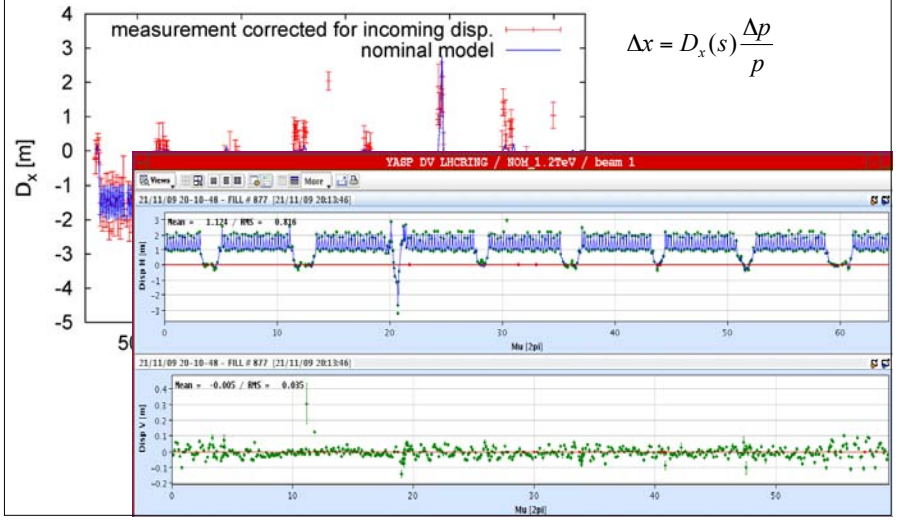
$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q}\int_{s1}^{s1+l}\beta(s_1)\Delta K\cos\left(2|\psi_{s1}-\psi_{s0}|-2\pi Q\right)ds$$

Δβ / β = 50 %



LHC Operation: the First Beam

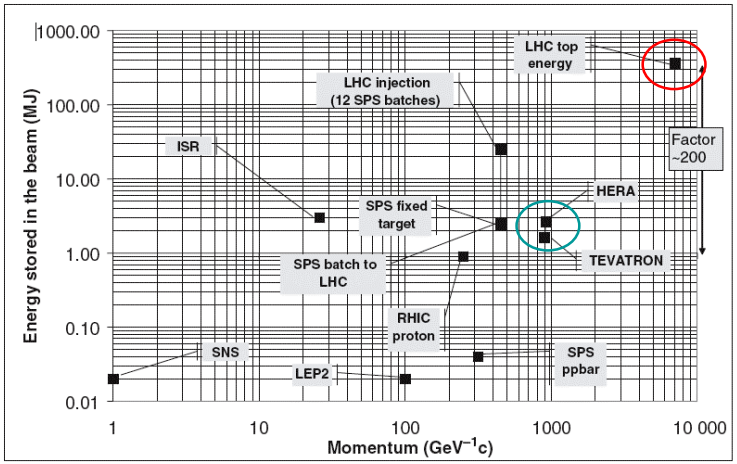
Dispersion Measurement



$$\Delta x = D_x(s)\frac{\Delta p}{p}$$

LHC Operation: Machine Protection & Safety

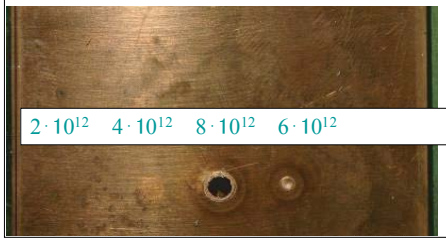
Energy Stored in the Beam of different Storage Rings



LHC Operation:
Machine Protection & Safety

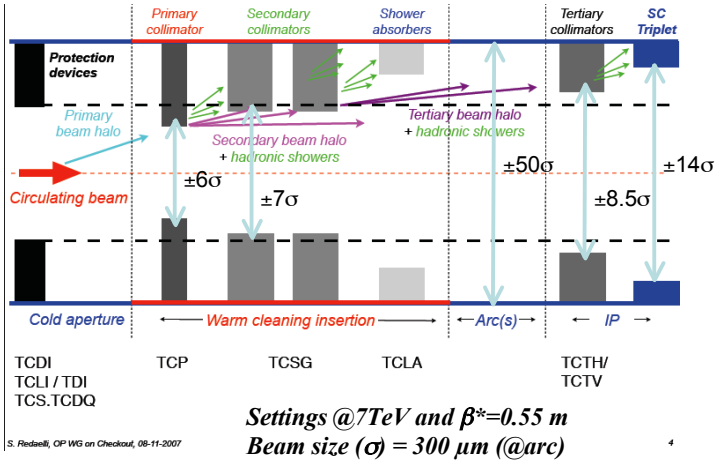
Energy stored in magnet system	10	GJ
Energy stored in one main dipole circuit	1.1	GJ
Energy stored in one beam	362	MJ

Enough to melt 500 kg of copper



$2 \cdot 10^{12}$	$4 \cdot 10^{12}$	$8 \cdot 10^{12}$	$6 \cdot 10^{12}$	450 GeV p Strahl

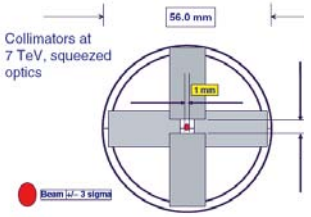
LHC Aperture and Collimation



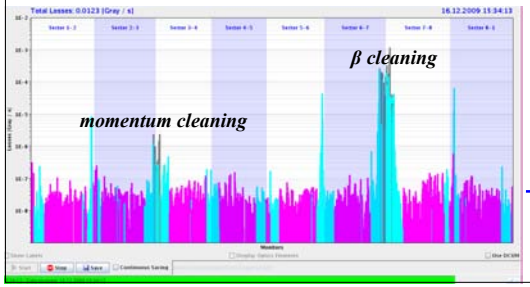
Settings @7TeV and β*=0.55 m
Beam size (σ) = 300 μm (@arc)
Beam size (σ) = 17 μm (@IR1, IR5)

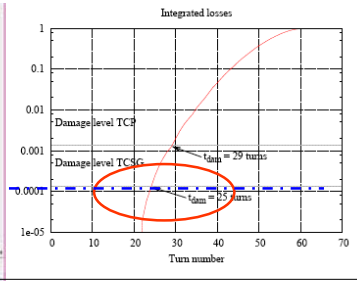
LHC Operation:
Machine Protection & Safety

... Komponenten des Machine Protection Systems :

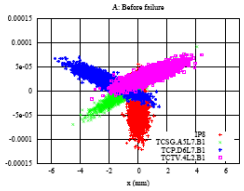


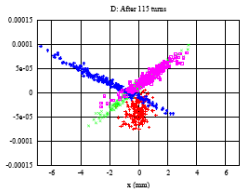
beam loss monitors
QPS
permit server
orbit control
power supply control
collimators
online on beam check of all (?)
hardware components
a fast dump
the gaussian beam profile





LHC Operation: Machine Protection & Safety





What will happen in case of Hardware Failure

Phase space deformation in case of failure of RQ4.LR7
(A. Gómez)

Short Summary of the studies:

quench in sc. arc dipoles: τ_{loss} ***=20 - 30 ms***
BLM system reacts in time, QPS is not fast enough

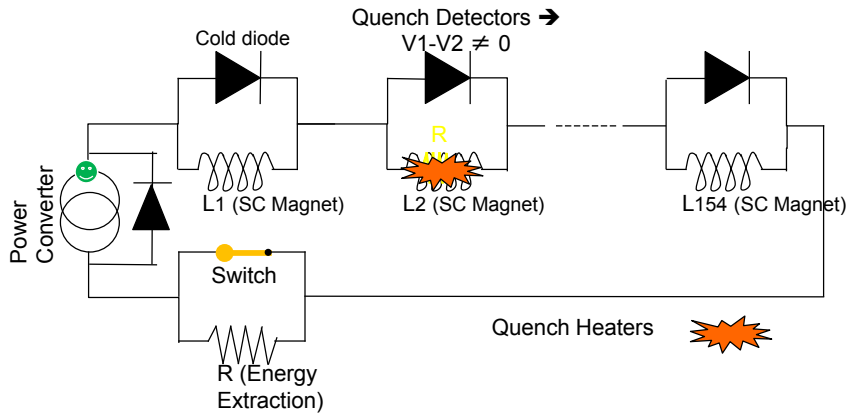
quench in sc. arc quadrupoles: τ_{loss} ***=200 ms***
BLM & QPS react in time

failure of nc. quadrupoles: τ_{det} ***= 6 ms*** ***→ FMCM installed***
τ_{damage} ***= 6.4 ms***

failure of nc. dipole: τ_{damage} ***= 2 ms***

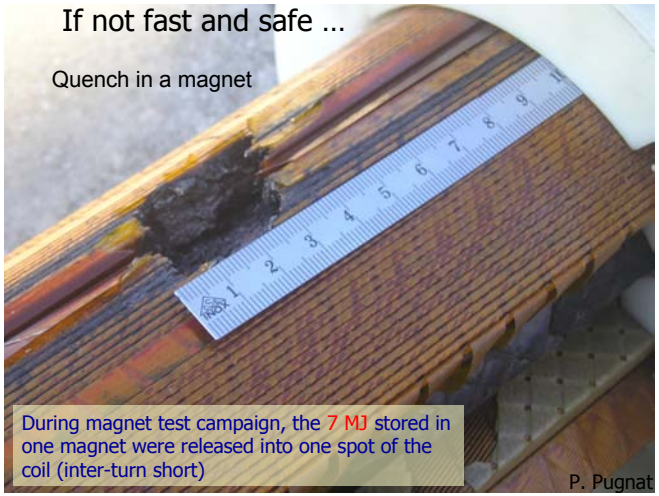
Energy stored in the magnets: 10 GJ
Quench Protection System

Schematics of the QPS in the main dipoles of a sector

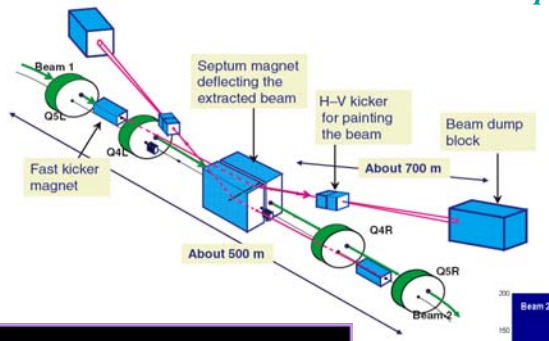


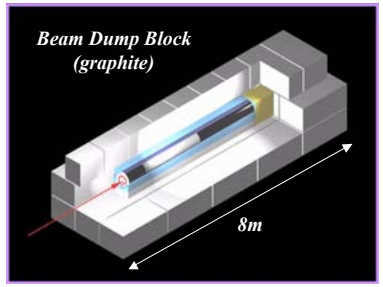
court. R. Alemany

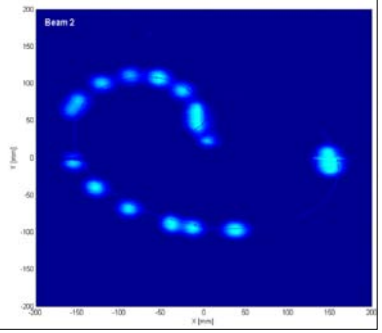
Energy stored in the magnets:
quench



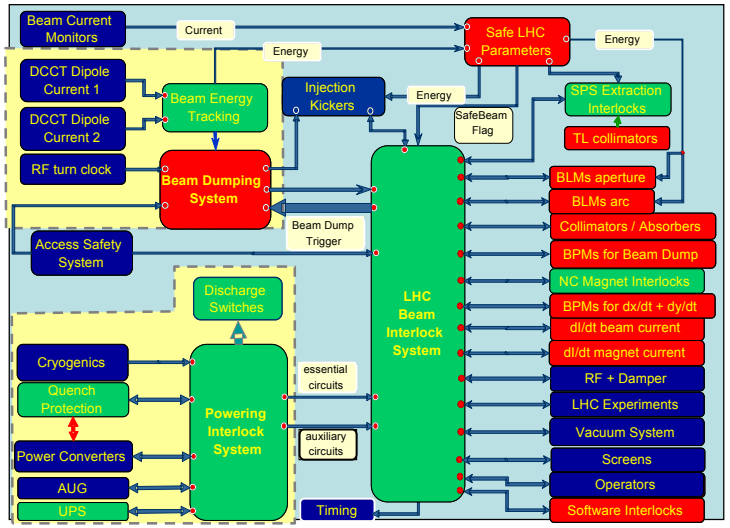
LHC Operation: Dump System







LHC Operation: Machine Protection & Safety



... no comment

Luminosity optimization

$$L = \frac{N_1 N_2 f_{rev} N_b}{2\pi\sqrt{\sigma_{1x}^2+\sigma_{2x}^2}\sqrt{\sigma_{1y}^2+\sigma_{2y}^2}} F \cdot W$$

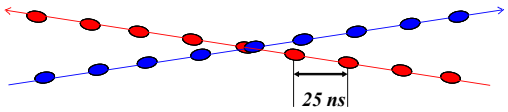
N_i = number of protons/bunch
N_b = number of bunches
f_{rev} = revolution frequency
σ_{ix} = beam size along x for beam i
σ_{iy} = beam size along y for beam i

F is a pure **crossing angle (*Φ*) contribution**:

$$F = \frac{1}{\sqrt{1+2\frac{\sigma_s^2}{\sigma_{1x}^2+\sigma_{2x}^2}\tan^2\frac{\phi}{2}}}$$

F_{LHC} = 0.836

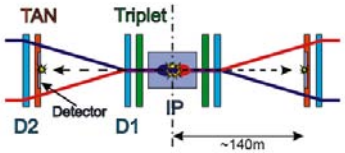
... cannot be avoided



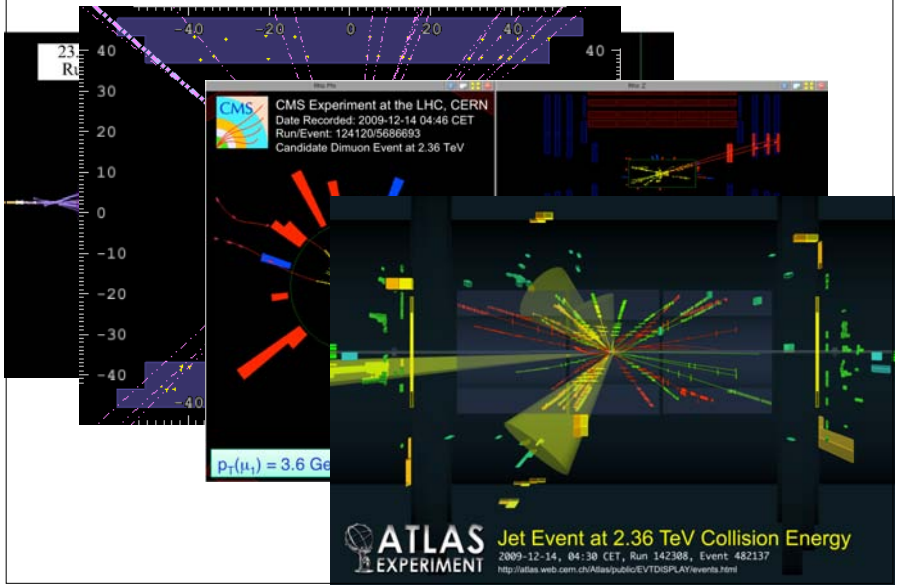
W is a **pure beam offset contribution**.

... can be avoided by careful tuning

$$W = e^{-\frac{(d_2-d_1)^2}{2(\sigma_{x1}^2+\sigma_{x2}^2)}}$$

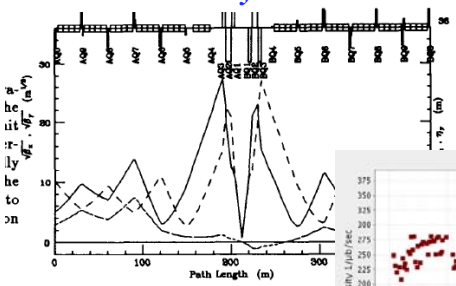


LHC Operation: the First Collisions at 2.36 TeV

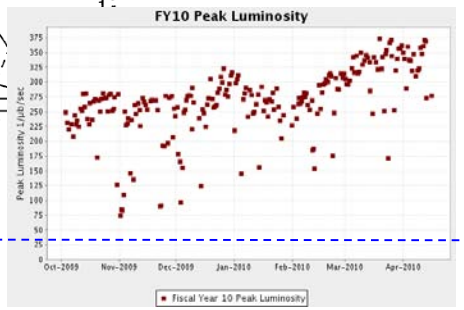


LHC Operation where are we ?

Tevatron Luminosity



... as in LHC: mirror symmetric triplet
$\beta^* = 0.35$ m Luminosity:
$L = 4*10^{32}$



LHC

LHC Operation where are we ?

	LHC Design	LHC 2010	Tevatron
Momentum at collision	7 TeV/c	3.5 TeV	1 TeV
Dipole field for 7 TeV	8.33 T	4.16 T	4.3 T
Protons per bunch	1.15×10^{11}	1.15×10^{11}	$2.7 / 1.0 \times 10^{11}$
Number of bunches/beam	2808	48	36
Nominal bunch spacing	25 ns	--	397 ns
Normalized emittance	3.75 µm	3.75 µm	3.0 µm
Absolute Emittance	5×10^{-10}	8×10^{-10}	2.8×10^{-9}
Beta Function	0.5 m	3.5 m	0.35 m
rms beam size (IP)	16 µm	53 µm	32 µm
Luminosity	1.0×10^{34}	2.0×10^{31}	4.0×10^{32}

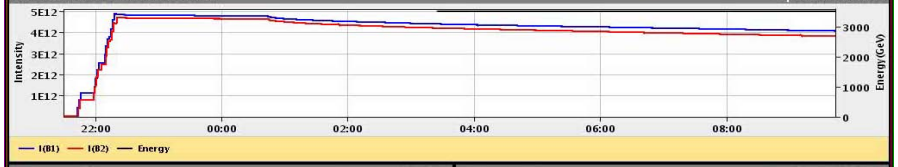
LHC Operation

23-Aug-2010 09:43:18 Fill #: 1298 Energy: 3500 GeV I(B1): 4.07e+12 I(B2): 3.82e+12

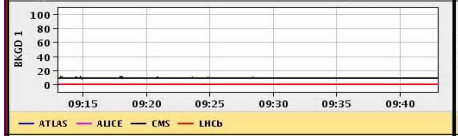
	ATLAS	ALICE	CMS	LHCb
Experiment Status	PHYSICS	PHYSICS	PHYSICS	PHYSICS
Instantaneous Lumi (ub.s)^-1	5.118	0.150	5.240	5.139
BRAN Luminosity (ub.s)^-1	4.917	0.162	4.592	4.521
Fill Lumiosity (nb)^-1	215.5	4.7	220.3	200.8
BKGD 1	0.020	0.017	9.304	0.207
BKGD 2	2.000	0.389	2.497	4.619
BKGD 3	0.000	0.006	0.003	0.087

LHCb VELO Position IN Gap: 0.0 mm | STABLE BEAMS | TOTEM: STANDBY

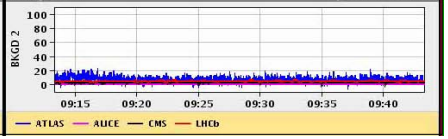
Performance over the last 12 Hrs Updated: 09:41:58



Background 1 Updated: 09:41:58

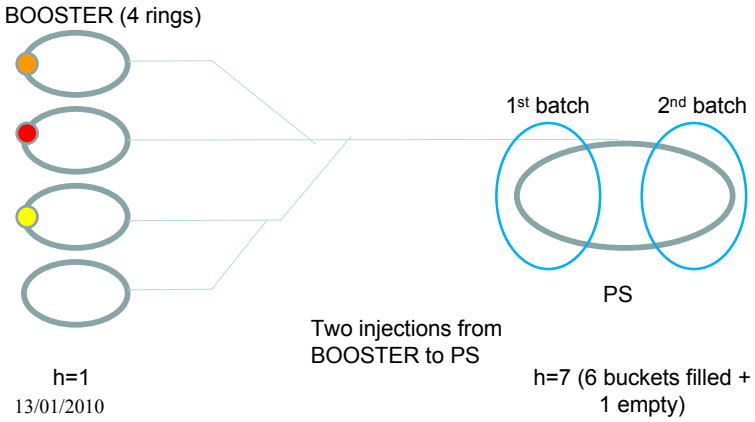


Background 2 Updated: 09:42:18



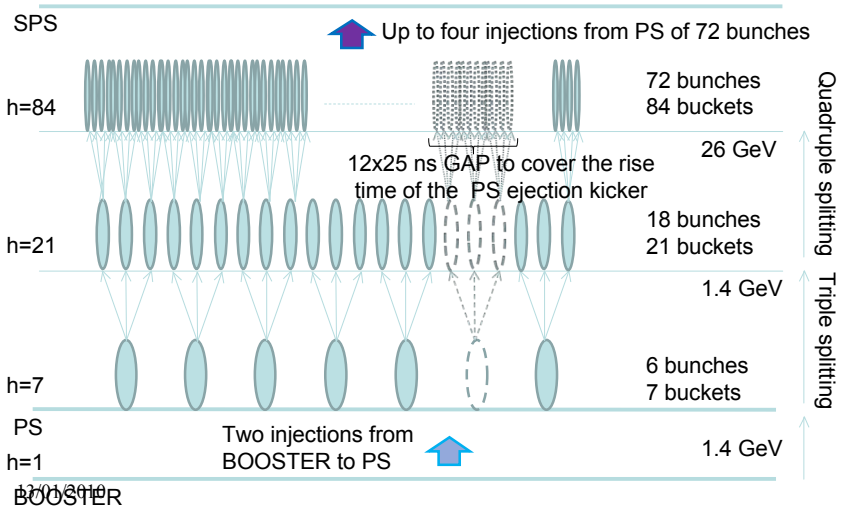
LHC Operation: Pre-Accelerators and Injection

BOOSTER (1.4 GeV) ➔ PS (26 GeV) ➔ SPS (450 GeV) ➔ LHC

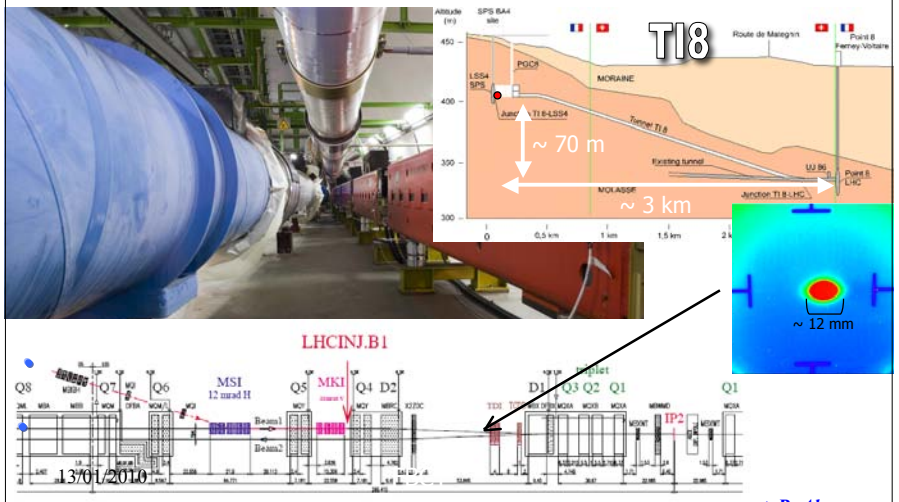


court. R. Alemany

LHC Injection: Preparing the Bunch Trains



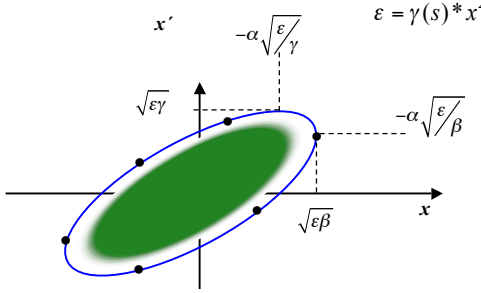
Injection mechanism: the transfer lines



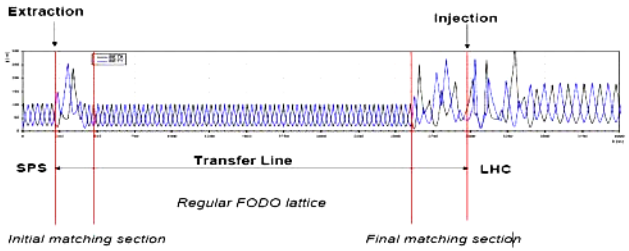
court. R. Alemany

LHC Injection: remember the phase space

$$\varepsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2$$



Injected Beam has to be matched to the optics of the storage ring



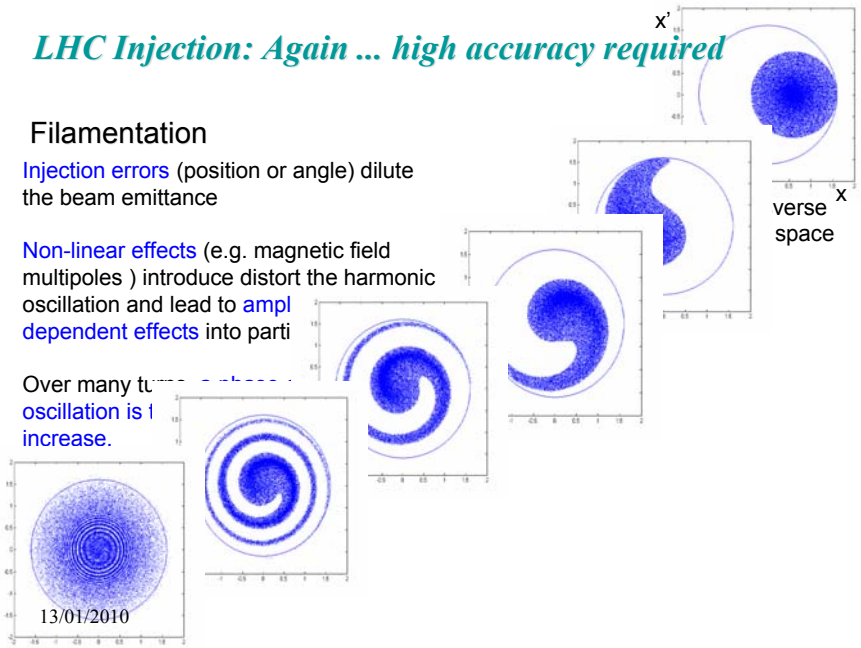
LHC Injection: Again ... high accuracy required

Filamentation

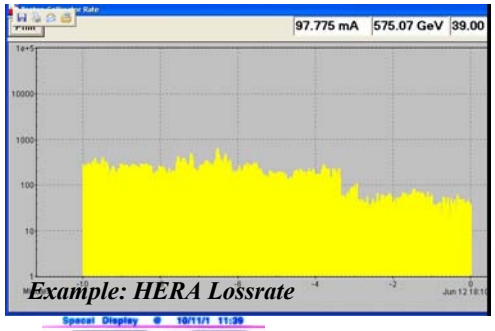
Injection errors (position or angle) dilute the beam emittance

Non-linear effects (e.g. magnetic field multipoles) introduce distort the harmonic oscillation and lead to ampl dependent effects into parti

Over many tu
oscillation is t
increase.



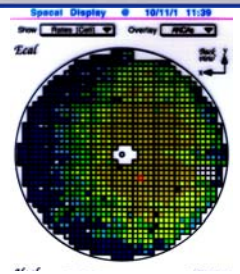
Collider "Luminosity Run": Background Optimisation



Example: HERA Lossrate

Optimise
- *Tune,*
- *Orbit,*
- *Chromaticity*
- *Coupling,*
- *Collision Point,*
- *Crossing Angle,*
- *Collimator Settings*

... to obtain optimum conditions



H1 Driftchamber Signals before ... and ... after Optimisation