Introduction to Quantum Field Theory and QCD

Lecture 3 & 4

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Lecture 3

- Before moving on to interacting theories there is a second type of field we want to learn how to quantise.
  - The Dirac field.
- Dirac Spinors.
- Quantising the Dirac Field.
- The Dirac Propagator.

Spin 0

- We have seen how to quantise the Klein-Gordon equation for a real scalar field.
- This describes particles of spin 0.
- The field is invariant under Lorentz transformations,

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

- In nature most particles carry spin and so we require fields that carry higher degrees of spin.

Dirac Equation

- The *Dirac field* will give rise to particles of spin 1/2.
- This field will transform differently under a Lorentz transformation. In general a field will transform as,

$$\phi^a(x) \to D\left[\Lambda\right]^a_b \phi^b(\Lambda^{-1}x) \qquad D[\lambda_1]D[\lambda_2] = D[\lambda_1\lambda_2]$$

- The equation of motion for the Dirac field is given by,

$$(i\gamma^\mu \partial_\mu - m)\,\Psi(x) = 0$$

- This equation is first order rather than second order as the KG equation is, but is still Lorentz invariant.

γ-Matrices

$$(i\gamma^\mu \partial_\mu - m)\,\Psi(x) = 0$$

- The γ-matrices are 4×4 matrices of the following form

The 2×2 identity matrix

$$\gamma_0 = \begin{pmatrix} 0 & \mathit{1} \\ \mathit{1} & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

- The σ_i are the 2×2 Pauli matrices.

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- They satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.

Anti-commutation relations

$\{A, B\} = AB + BA$

Properties of γ-Matrices

- The γ-matrices satisfy the Clifford Algebra.

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbf{1}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- When we have $\mu \neq \nu$ then

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \qquad (\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1$$

- There are many different possible representations of the gamma matrices, they are all equivalent via unitarity transforms.

- The representation we will use is known as the *Weyl* representation.

Dirac Spinors

$$(i\gamma^\mu \partial_\mu - m)\,\Psi(x) = 0$$

- The Dirac equation describes the motion of a Spinor field Ψ.
- Spinors do not transform as Lorentz scalars, instead they transform under the $\Lambda_{1/2}$ representation of the Lorentz group,

$$\Psi(x) \to \Lambda_{1/2}\Psi(\Lambda^{-1}x)$$

- Where Λ is the Lorentz transformation of a 4-vector,

$$x^\mu \to \Lambda^\mu_\nu x^\nu$$

- The gamma matrices transform as,

$$\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu_\nu\gamma^\nu$$

Lorentz Transformations

- It is straightforward to see that the Dirac equation is invariant under Lorentz transformations,

$$(i\gamma^\mu \partial_\mu - m)\,\Psi(x) = 0$$

- Lagrangians are also invariant,

$$\mathcal{L}(x_\mu) \to \mathcal{L}((\Lambda^{-1})^\nu_\mu x_\nu)$$

- As Lagrangian's are invariant under Lorentz transformations then we need to work out how to write down *products of Dirac spinors* that form Lorentz scalars.

First Attempt at a Spinor Product

- Our first guess might be $\Psi^\dagger\Psi$.
- This does not work as it transforms as,

$$\Psi^\dagger\Lambda_{1/2}^\dagger\Lambda_{1/2}\Psi$$

- $\Lambda_{1/2}$ is not Unitary,

$$\Lambda_{1/2}^\dagger\Lambda_{1/2} \neq 1$$

- So $\Lambda_{1/2}^\dagger \neq \Lambda_{1/2}^{-1}$ and $\Psi^\dagger\Psi$ does not transform as a scalar.

Second Attempt at a Spinor Product

- We need to consider a slightly more complicated expression.
- If we use γ_0 we can define the quantity, which we call the *Dirac Adjoint*,

$$\overline{\Psi} = \Psi^\dagger \gamma^0 \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Due to the properties of γ_0, this transforms as,

$$\overline{\Psi} \rightarrow \overline{\Psi} \Lambda^{-1}_{1/2}$$

- We can then define the following Lorentz invariant bilinear,

$$\overline{\Psi}\Psi$$

Other Bilinear Covariants

- We can build up other objects as well,

$$\overline{\Psi}\gamma^{\mu}\Psi$$

- This transforms as Lorentz vector,

$$\overline{\Psi}\gamma^{\mu}\Psi \to \Lambda^{\mu}_{\nu}\overline{\Psi}(\Lambda^{-1}x)\gamma^{\nu}\Psi(\Lambda^{-1}x)$$

- So we can treat the gamma matrices as 4-vectors. Contracting it with 4-vectors gives Lorentz scalars.
- Similarly we can write down an object that transforms as a Lorentz tensor,

$$\overline{\Psi}\gamma^{\mu}\gamma^{\nu}\Psi$$

The Dirac Lagrangian

- It is now possible to build up a Lagrangian for the Dirac theory using these Bilinear objects,

$$\mathcal{L}_{\mathrm{Dirac}} = \overline{\Psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi$$

- From which the Dirac equation, is readily reproduced using the Euler-Lagrange equation with $\overline{\Psi}$.

- The Dirac spinor is a four component object, how do we go about writing these components down?

Spinor Representations

- Our starting point is to note that the Dirac representation of the spinors is reducible.
- The four component Dirac spinor can be written as two 2-component spinors,

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$$

- These are the left-handed, Ψ_L, and right-handed, Ψ_R, *Weyl spinors*.

Rewriting the Dirac Eq

- Using this representation of the spinors we can rewrite the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\Psi = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix} \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0$$

- If we set m=0 then we see that these massless spinors decouple into two equations, one for the left-handed spinors and one for the right-handed spinors,

$$i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\Psi_L = i\sigma\cdot\partial\Psi_L = 0 \qquad \sigma = (1, \vec{\sigma})$$

$$i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\Psi_R = i\overline{\sigma}\cdot\partial\Psi_R = 0 \qquad \overline{\sigma} = (1, -\vec{\sigma})$$

γ_5

- For our particular choice of the gamma matrices we produced a Chiral representation of the spinors.

- To do this for any choice of representation we can take advantage of a useful object,

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

- This satisfies the relations,

$$\{\gamma_5, \gamma_\mu\} = 0 \qquad (\gamma_5)^2 = 1$$

- In the Weyl representation γ_5 is given by,

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Helicity Projection

- Using this we can construct a projection operator,

$$P_{\pm} = \frac{1}{2}(1 \pm \gamma^5)$$

- This satisfies,

$$P_{\pm}^2 = P_{\pm} \qquad P_+ P_- = 0$$

- We can then define the chiral states as,

$$\Psi_{\pm} = P_{\pm}\Psi$$

$$\Psi_+ = \Psi_R$$
$$\Psi_- = \Psi_L$$

Notation

- There is an alternative notation that we will find useful later on,

$$\langle p^- | = \Psi_-, \quad \langle p^+ | = \Psi_+$$

- When writing down combinations of gamma matrices contracted with 4-vectors there is the compact notation

$$\not{a} = \gamma^\mu a_\mu$$

Solving the Dirac Equation

- Next we want to write down forms for the spinors that satisfy the Dirac equation.
- To do this start with an ansatz for the solution,

$$\Psi = u(\vec{p})e^{-ip\cdot x}$$

- $u(\vec{p})$ is a four component spinor and all the space time dependance is now in the exponential.
- The Dirac equation can then be written as,

$$(\not{p} - m)u(\vec{p}) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\overline{\sigma}^\mu & -m \end{pmatrix} u(\vec{p}) = 0$$

Solutions of the Dirac Equation

- Using $(p\cdot\sigma)(p\cdot\overline{\sigma}) = m^2$ we can then write the positive frequency solution $u(\vec{p})$,
$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\overline{\sigma}}\xi \end{pmatrix}$$
- Here ξ is a two component object normalised so that $\xi^\dagger\xi = 1$.
- Similarly there is a a negative frequency solution for, $\Psi = v(\vec{p})e^{ip\cdot x}$, which satisfies the Dirac equation,

$$(\not{p} + m)v(\vec{p}) = \begin{pmatrix} m & p_\mu\sigma^\mu \\ p_\mu\overline{\sigma}^\mu & m \end{pmatrix} v(\vec{p}) = 0 \Rightarrow v(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta \\ -\sqrt{p\cdot\overline{\sigma}}\eta \end{pmatrix}$$

Helicity

- The helicity is the projection of the angular momentum in the direction of momentum.
- For massless particles this will be the same as the spin.
- We can compute the helicity of the particle using the operator,

$$h = \frac{1}{2}\frac{k_i}{|\vec{k}|}\left(\begin{array}{cc} \sigma^i & 0 \\ 0 & \sigma^i \end{array}\right)$$

- This acting on the positive/negative massless chiral solutions to the Dirac equation will give ±(1/2).

Spinor Products

- Write these spinor solutions u and v in terms of a component basis for the ξ^s and η^s, for example
$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
- We can then write down a spinor inner product,

$$u^r(\vec{p}) \cdot u^s(\vec{p}) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right) \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}$$
$$= \xi^{r\dagger}(p\cdot\sigma)\xi^s + \xi^{r\dagger}(p\cdot\bar{\sigma})\xi^s = 2\xi^{r\dagger}p_0\xi^s = 2p_0\delta^{rs}$$

Spinor Products

- As well as

$$\overline{u}^r(\vec{p}) \cdot u^s(\vec{p}) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\overline{\sigma}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\xi^s \end{pmatrix}$$
$$= 2m\delta^{rs}$$

- There are similar result for the v spinors.
- The remaining spinor inner products of u and v give zero.
- The spinor outer product is,

$$\sum_{s=1}^{2} u^s(\vec{p})\overline{u}^s(\vec{p}) = \not{p} + m \qquad \sum_{s=1}^{2} v^s(\vec{p})\overline{v}^s(\vec{p}) = \not{p} - m$$

Quantising the Dirac field

- We want to quantise the Dirac Lagrangian,

$$\mathcal{L} = \overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi$$

- We will proceed step by step as we did for the scalar field.
- First we note that the conjugate momentum is simply, $\pi_\Psi = i\Psi^\dagger$, because the Dirac equation is a first order equation.

Quantising the Dirac field

- Our next step would be to promote the field Ψ and its conjugate momentum $i\Psi^\dagger$ to operators.
- Taking out inspiration from the scalar case we assume the form of the operator Ψ can be written down in terms of ladder operators (in the Schrodinger picture),

$$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \sum_{s=1,2} \left(a^s(\vec{p}) u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^{s\dagger}(\vec{p}) v^s e^{-\vec{p}\cdot\vec{x}} \right)$$

$$\Psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \sum_{s=1,2} \left(a^{s\dagger}(\vec{p}) u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^s(\vec{p}) v^s e^{-\vec{p}\cdot\vec{x}} \right)$$

Ladder Operators

- There are two ladder operators here
  - $a^s(\vec{p})$ associated with the positive frequency states $u^s(\vec{p})$
  - $b^s(\vec{p})$ associated with the negative frequency states $v^s(\vec{p})$
- As before the vacuum is defined to be,

$$a(\vec{p})^s|0\rangle = 0|0\rangle = 0, \quad b(\vec{p})^s|0\rangle = 0|0\rangle = 0$$

Commutation Relations?

- The next step would then be to impose equal time commutation relations on Ψ and $\Psi^\dagger$

$$[\Psi_a(\vec{x}), \Psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x} - \vec{y})\delta_{ab}$$

- The ladder operators would then satisfy corresponding commutation relations to satisfy these.
- Spinors are fermions and so obey Fermi-Dirac statistics.
- Dirac fields anti-commute rather than commute.
- Commutation relations do not correctly quantise the theory, we would would end up with negative energy states among other problems.

Anti-commutation Relations

- As the Dirac field anti-commutes then we will instead impose anti-commutation relations on the operators Ψ and $\Psi^\dagger$,

$$\{\Psi_a, \Psi_b^\dagger\} = \delta^{(3)}(\vec{x} - \vec{y})\delta_{ab}, \qquad \{\Psi_a, \Psi_b\} = \{\Psi_a^\dagger, \Psi_b^\dagger\} = 0$$

- Similarly the ladder operators must also now satisfy anti-commutation relations,

$$\{a_r(\vec{p}), a_s^\dagger(\vec{q})\} = (2\pi)^3 \delta_{rs} \delta^{(3)}(\vec{p} - \vec{q}),$$

$$\{b_r(\vec{p}), b_s^\dagger(\vec{q})\} = (2\pi)^3 \delta_{rs} \delta^{(3)}(\vec{p} - \vec{q}),$$

The Hilbert Space

- We can show that the Ladder creation and destruction operators for the different types of spinor satisfy commutation relations with the Hamiltonian,

$$[H, b^r(\vec{p})] = -\omega(\vec{p})b^r(\vec{p}) \quad [H, b^{r\dagger}(\vec{p})] = \omega(\vec{p})b^{r\dagger}(\vec{p})$$
$$[H, a^r(\vec{p})] = -\omega(\vec{p})a^r(\vec{p}) \quad [H, a^{r\dagger}(\vec{p})] = \omega(\vec{p})a^{r\dagger}(\vec{p})$$

- We can therefore use them to build up the Hilbert Space of states as before,

$$|\vec{p}, r\rangle = b^{r\dagger}(\vec{p})|0\rangle$$

- From this we can see the Fermi-Dirac nature of the states,

$$|p_1, p_2\rangle = a^\dagger(\vec{p}_1)a^\dagger(\vec{p}_2)|0\rangle = -a^\dagger(\vec{p}_2)a^\dagger(\vec{p}_1)|0\rangle = -|p_2, p_1\rangle$$

- If we were to try to create two states of the same momentum we would have,

$$a_r^\dagger(\vec{p})a_s^\dagger(\vec{p})|0\rangle \equiv -a_s^\dagger(\vec{p})a_r^\dagger(\vec{p})|0\rangle = 0$$

Dirac Hamiltonian

- The Hamiltonian of the Dirac Theory is given by,

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \omega(\vec{p}) \left(a_s^\dagger(\vec{p}) a_s(\vec{p}) + b_s^\dagger(\vec{p}) b_s(\vec{p}) \right)$$

- This shows that all states produced have positive energy, as we desire.
- If we had carried on computing with commutation relations we would have a negative sign, allowing us to create an infinite number of negative energy states.
- This result shows a defining feature of these theories, *we must use anti-commutation relation for Fermions and commutation relations for Bosons*. If we want to preserve Lorentz invariance and causality as well as have only positive energies and norms.

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Dirac Propagator

- Now that we have quantised the spinor field we want to write down its propagator.
- In the scalar case the propagator was derived from the commutator of two fields, in analogy we investigate the anti-commutator relations between two fields to get the spinor propagator.
- We wish to write down the Feynman propagator so we will start from the time ordered two particle correlation function,

$$S_F(x-y) = \langle 0|T\{\Psi(x)\overline{\Psi}(y)\}|0\rangle$$

Dirac Propagator

- As we are working with anti-commuting fields the Time ordering operation inserts a minus sign when we swap the order of the fields,

$$\langle 0|T\{\overline{\Psi}(y)\Psi(x)\}|0\rangle = \begin{cases} \langle 0|\Psi(x)\overline{\Psi}(y)|0\rangle & x^0 > y^0 \\ \langle 0| -\overline{\Psi}(y)\Psi(x)|0\rangle & x^0 < y^0 \end{cases}$$

- These correlation functions can be written as,

$$\begin{aligned} \langle 0|\Psi_a(x)\overline{\Psi}_b(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\vec{p})}\sum_s u_a^s(p)\overline{u}_b^s(p)e^{-ip\cdot(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\vec{p})}(\not{p}+m)e^{-ip\cdot(x-y)} \\ &= (i\not{\partial}_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\vec{p})}e^{-ip\cdot(x-y)} \end{aligned}$$

Dirac Propagator

- And for the other time ordering

$$\langle 0| - \overline{\Phi}_b(y)\Phi_a(x)|0\rangle = (i\partial\!\!\!/_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\vec{p})} e^{ip\cdot(x-y)}$$

- The expression inside the integral is just that of a real scalar propagator D(x-y).
- Acting on this with the $i(\partial\!\!\!/_x + m)$ operator gives the Dirac Feynman propagator,

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon}$$

Dirac Propagator

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$$\langle 0| - \overline{\Phi}_b(y)\Phi_a(x)|0\rangle = (i\partial\!\!\!/_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega(\vec{p})}e^{ip\cdot(x-y)}$$

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Summary

- We now know about Dirac Spinors their Lorentz transformation properties and how to write down solutions of the Dirac equation in terms of them.
- We can now Quantise the Dirac Field.
- We have derived the Dirac Propagator.

Lecture 4

- Introduce Interacting theories.
- The Interaction picture.
- Relate full fields to free fields.
- Wick's Theorem.

Interactions

- So far we have quantised non-interacting theories and derived their propagators.
- To compare against the real world we need to deal with interacting theories.
- The free particle states are no longer eigenstates of the Hamiltonian.
- This will make our lives more difficult as we cannot directly apply the simple procedure we used before.
- As a first step we will find it useful to split the Lagrangian into two pieces,

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{int}}$$

Splitting the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{int}}$$

- $\mathcal{L}_{\text{kinetic}}$ contains all the non-interacting, kinetic terms.
- $\mathcal{L}_{\text{int}}$ contains all the interacting terms.
- Typically these are non-linear combinations of local fields, i.e. terms of the type $\phi(x)\phi(x)\phi(x)$
- We will use as our main example ϕ^4theory,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

Example Theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

- The final term is the interaction term and has a coupling constant of λ.
- To perform computations within this theory we cannot proceed directly as in the case of the free field.
- To see this let us start with the simplest object we could compute, the two-point time ordered correlation function,

$$\langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle$$

Interacting Fields

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$$

- In the free theory this object is the propagator.
- The ground state of the interacting theory $|\Omega\rangle$ is different from that of the free theory $|0\rangle$.
- We can no longer solve the theory for all time by directly using ladder operators and the analogy with SHO's.
- The full theory is too hard to solve completely we will instead take the tactic of computing what we can by relating it to the free theory.
- We will compute a perturbative expansion.

Evolving Interacting States

- Using the ladder operators we can write the scalar field Φ at a fixed time t_0 as

$$\phi(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} \left(a(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)$$

- To obtain this operator at different values of t we switch to the Heisenberg picture,

$$\phi(x) = e^{iHt} \phi(\vec{x}) e^{-iHt}$$

- Unlike for the free case we cannot just commute e^{iHt} through Φ as

$$H = H_0 + H_{int}$$

- The Hamiltonian, H, is now made up of two parts, H_{int} is the Hamiltonian of the interacting term and H_0 of the free field i.e. something we can solve.

The Interaction Picture

- To proceed we set the interaction term to zero, λ=0.
- We can then bring the exponential through Φ to get the time dependance of the field in this limit,

$$\phi(x)|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)} \equiv \phi_I(x)$$

- We call $\phi_I(x)$ the field in the *interaction picture*.
- When λ is small this will still give the most important part of the time dependance of $\phi(x)$.

The Interaction Picture

- This is a halfway point between the Schrodinger picture and the Heisenberg picture, as now both states and operators evolve with time.
- In the interaction picture states evolve in time according to the interaction Hamiltonian,

$$|\phi(t)\rangle_I = e^{iH_0 t}|\phi(t)\rangle_S$$

- Operators also evolve in time according to the free Hamiltonian.

$$O_I(t) = e^{iH_0 t} O_S e^{-iH_0 t}$$

Evolution operator

- So far this seems too limited, how do we relate the interaction picture fields to the full fields?
- Translate the interaction picture field into the Heisenberg picture of the full field Φ,

$$\begin{aligned}\phi(x) &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(x) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \\ &= U^\dagger(t,t_0)\phi_I(x)U(t,t_0)\end{aligned}$$

- This relates the full fields to the interaction picture fields via an *evolution* operator U,

$$U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

- The U operators are unitary and satisfy

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$$

Evolution Operators

- For this to be useful we need to express U in terms of the interaction picture fields ϕ_I.
- This can be done by noting that $\mathrm{U(t,t_0)}$ with the initial condition $\mathrm{U(t,t_0)=1}$ satisfies the Schrodinger equation,

$$
\begin{aligned}
i\frac{\partial}{\partial t}U(t,t_0) &= e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\
&= e^{iH_0(t-t_0)}(H_{\text{int}})e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\
&= H_I(t)U(t,t_0)
\end{aligned}
$$

- Where $H_I(t)$ is the interaction Hamiltonian in the interaction picture, and is given by,

$$H_I = e^{iH_0(t-t_0)}(H_{\text{int}})e^{-iH_0(t-t_0)}$$

Solving the Evolution Operator

$$i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)U(t,t_0)$$

- We now want to solve this equation.
- Doing this will lead to a perturbative expansion.
- We start from,

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)U(t_1,t_0)$$

- Then we iterate this solution,

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)\left(1 + (-i)\int_{t_0}^{t_1} dt_2 H_I(t_2)U(t_2,t_0)\right)$$

Solving the Evolution Operator

- If we keep on doing this we get a perturbative expansion,

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)$$
$$+(-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2)$$
$$+(-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1) H_I(t_2) H_I(t_3) + \ldots .$$

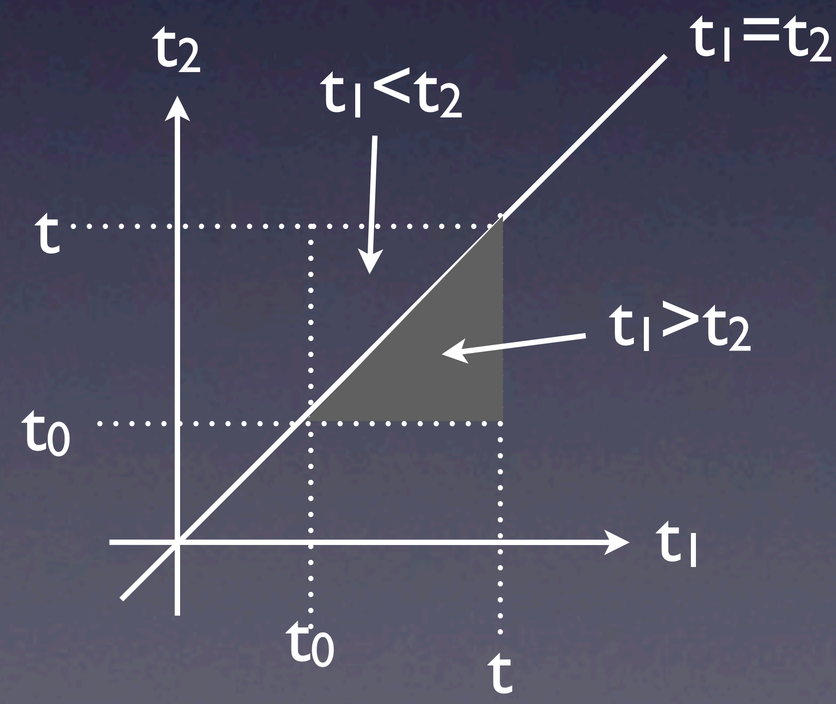
- With each time in order $t>t_1>t_2>\ldots>t_0$

Simplifying the Time Ordering

- We note that,

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\{H_I(t_1)H_I(t_2)\}$$

- This can be visualised,



Time Ordered Result

- As each field stands in time order then we can rewrite this in a more compact form,

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)$$
$$+\frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 T\left\{H_I(t_1)H_I(t_2)\right\}$$
$$+\frac{(-i)^3}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 dt_3 T\left\{H_I(t_1)H_I(t_2)H_I(t_3)\right\} + \dots$$

- This can further be rewritten in the more compact notation,

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt' H_I(t')\right]\right\}$$

- This time ordered exponential is defined as the time-ordering of each term in the Taylor series.

The Ground State

- Now that we have expressed the field $\phi(x)$ as a perturbative expansion in terms of the free field we must now do something similar for the ground state of the interacting theory $|\Omega\rangle$.
- This can be done using the evolution operator U,

$$|\Omega\rangle = U(t_0, -T)|0\rangle$$

- The state evolves from the ground state of the free theory at some time $-\mathrm{T}$ to a time t_0.
- We will eventually want to take T to be the infinite past.

Rewriting Correlation Functions

- Take the full field expressed in terms of fields in the interaction picture and the ground state written in terms of this as well gives,

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)}$$

$$\frac{\langle 0|T\{U(T,x_1)\phi(x_1)U(x_1,x_2)\phi(x_2)U(x_2,x_3)\ldots U(x_{n-1},x_n)\phi(x_n)U(x_n,-T)\}|0\rangle}{\langle 0|T\{U(T,-T)\}|0\rangle}$$

- The denominator factor is to ensure the correct normalisation, we will see the role this term has a little bit later on.

Final Form

- Making use of the Time ordering operator we have our final compact form for a correlation function of our full theory (where we have dropped the I label for the interaction picture fields)

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\exp\left[-i\int_T^{-T} dt H_I(t)\right]\}|0\rangle}{\langle 0|T\{\exp\left[-i\int_T^{-T} dt H_I(t)\right]\}|0\rangle}$$

Computing Correlation Functions

- The problem of computing with the interaction theory has been reduced to calculating time ordered products of interaction picture fields,

$$\langle 0| T\left\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\right\}|0\rangle$$

- When we only have two fields this reduces to simply the propagator of the free theory.
- What about more general cases?
  - We could expand out the Φ_I's in terms of ladder operators and then compute from there.
  - There is a better approach.

Wick's Theorem

- Instead we can apply *Wick's theorem* to simplify the amount of computation involved,

$$T\left\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\right\} =: \phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n) :$$

$$+ : \text{all possible contractions} :$$

- We define a *contraction* of the fields $\Phi(x_1)$ and $\Phi(x_2)$ to be the Feynman propagator of these fields $D_F(x_1 - x_2)$.
- The N operator means the fields are normally ordered, rather than time ordered.

Field Components

- To understand this better let us break up the field into two pieces

$$\phi(x) = \phi^+(x) + \phi^-(x)$$
$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} a(\vec{p}) e^{-ip\cdot x}$$
$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\vec{p})}} a^\dagger(\vec{p}) e^{+ip\cdot x}$$

- Rewrite the time ordered product of fields.
  - When $x_0 > y_0$ we have

$$T\phi(x)\phi(y) = \phi(x)\phi(y)$$
$$= \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \left[\phi^+(x), \phi^-(y)\right] + \phi^-(x)\phi^-(y)$$

Normal Ordering

$$\phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \left[\phi^+(x), \phi^-(y)\right] + \phi^-(x)\phi^-(y)$$

- This expression is now normally ordered, i.e. all "+" states are ordered before all "-" states. We denote this using the $::$ notation.

- Also we have picked up a commutator which is simply part of the propagator. For the time ordering $x_0>y_0$ we rewrite this expression as

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +D(x-y)$$

- For the other time ordering $y_0>x_0$ we get

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +D(x-y)$$

- Combining the two we get the simplest example of the general relationship,

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +D_F(x-y)$$

Contractions

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D_F(x-y)$$

- A contraction is the commutator of two components of the field.
- This can then be related to a propagator.
- A second example is the 4 field correlation function

$$T\left\{\phi_1\phi_2\phi_3\phi_4\right\} =: \phi_1\phi_2\phi_3\phi_4 : + : D(x_1-x_2)\phi_3\phi_4 : + : D(x_1-x_3)\phi_2\phi_4 :$$

$$+ : D(x_1-x_4)\phi_2\phi_3 : + : D(x_2-x_3)\phi_1\phi_4 : + : D(x_2-x_4)\phi_1\phi_3 :$$

$$+ : D(x_3-x_4)\phi_1\phi_2 : + D(x_1-x_2)D(x_3-x_4)$$

$$+ D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3)$$

Simpler Computation

- How has changing the ordering helped us?
- The Normal ordering operation moves all destructive ladder operators to the right. These will then annihilate against the ground state,

$$:a_1 a_2^\dagger a_3:|0\rangle =: a_2^\dagger a_1 a_3:|0\rangle = 0$$

- Any operator which is not fully contracted will then vanish.
- This means that only the propagator terms will survive inside the correlation function.

Approaching Feynman Diagrams

- We have reduced the computation of a correlation function of a product of fields to the computation of the set of all ways of connecting the fields via propagators.
- In the four field case we have,

$$T\left\{\phi_1\phi_2\phi_3\phi_4\right\} = D(x_1 - x_2)D(x_3 - x_4)$$

$$+D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3)$$

- This is beginning to look very much like Feynman diagrams as we can interpret these terms in a diagrammatic way.

Summary

- Introduce Interacting theories.
- Switched into the Interaction picture, to relate the full fields of the theory which we cannot compute to the free fields we can compute.
- We have seen that this leads to a perturbative expansion.
- We have the beginnings of a diagrammatic approach to computation through the use of Wick's Theorem.