Introduction to Quantum Field Theory and QCD

Lecture 5 & 6

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Lecture 5

- How to write down Feynman Diagrams.
- *In* and *out* states and their relation to the S-Matrix.
- Computing cross sections.

Computing with Diagrams

$$T\left\{\phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n)\right\} =: \phi_I(x_1)\phi_I(x_2)\ldots\phi_I(x_n) :$$

$$+ : \text{all possible contractions} :$$

- Wick's Theorem greatly simplifies the computation of Correlation functions of fields.
- We still need to compute all possible complete contractions of the fields.
- Each contraction is a propagator which connects two fields.
- Interpret the set of possible propagators as a set of diagrams.

A Simple Example

$$T\{\phi_1\phi_2\phi_3\phi_4\} = D(x_1 - x_2)D(x_3 - x_4)$$
$$+ D(x_1 - x_3)D(x_2 - x_4) + D(x_1 - x_4)D(x_2 - x_3)$$

- If we were to interpret the four field example as a set of diagrams we see that we have three terms each of which involves a field that evolves from one point to another,

- Instead of working these out using Wick's Theorem we could simply have written them down.

Including Interactions

- The previous example shows what happens in a non-interacting free theory, the points merely evolve in time without interfering with each other.
- What about interactions?
- Consider the second term in the perturbative expansion of the two point correlation function (the first term just being the free field result)

$$\langle 0|T\{\phi(x)\phi(y)\exp\left[-i\int_{t_0}^{t} dt' H_I(t')\right]\}|0\rangle$$

$$= \langle 0|T\{\phi(x)\phi(y)1|0\rangle + \langle 0|T\{\phi(x)\phi(y)(-i)\int_{t_0}^{t} dt' H_I(t')\}|0\rangle \text{ +...}$$

Wick's Theorem

- After substituting our interaction Hamiltonian, $H_{\text{int}} = -\frac{\lambda}{4!}\phi^4(x)$ we have,

$$\langle 0|\phi(x)\phi(y)(-i)\int d^4z\frac{\lambda}{4!}\phi(z)\phi(z)\phi(z)\phi(z)|0\rangle$$

- Applying Wick's Theorem to this expression we get,

$$3\left(\frac{-i\lambda}{4!}\right)D_F(x-y)\int d^4zD_F(z-z)D_F(z-z)$$

$$+12\left(\frac{-i\lambda}{4!}\right)\int d^4zD_F(x-z)D_F(y-z)D_F(z-z)$$

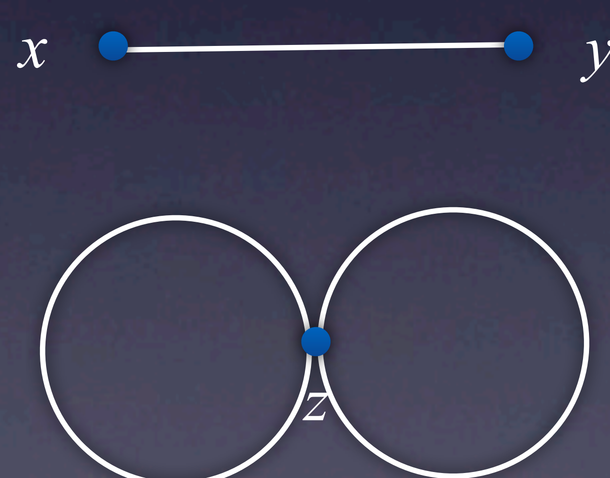
- The integers in front of the two expressions are due to the number of contractions that give equivalent results.

Diagrams

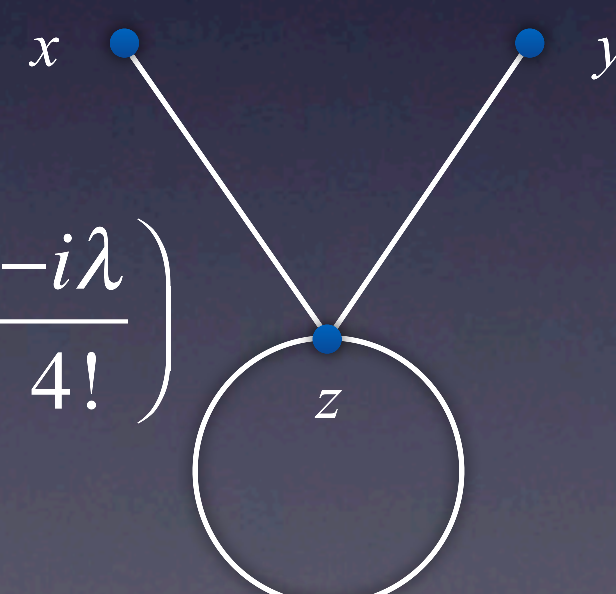
$$3\left(\frac{-i\lambda}{4!}\right) D_F(x-y)\int d^4z D_F(z-z) D_F(z-z)$$
$$+12\left(\frac{-i\lambda}{4!}\right)\int d^4z D_F(x-z) D_F(y-z) D_F(z-z)$$

- This can be written down as two diagrams

$\int d^4z\left(\frac{-i\lambda}{4!}\right)$



$\int d^4z\left(\frac{-i\lambda}{4!}\right)$



The Interaction Vertex

- Each interaction vertex has four fields at the same point.
- Each has an associated factor of $\int d^4z(-i\lambda)$
- The 4! in the denominator is cancelled by 4! coming from the different ways of arranging the four legs of the vertex.
- Each such vertex can be interpreted as the emission and/or absorption of particles at the vertex (from the four fields at the same point) summed over all points where this process can occur (from the integral over z).
- We must also take into account the symmetry factor of the diagram which is basically the number of ways of interchanging the components in the diagram with out changing the diagram itself.

Momentum Space

- We are almost ready to write down the Feynman rules for this theory.
- Ideally we want to deal with these rules in terms of momenta and not space-time positions.
- To do this insert the expression for the propagator in the form,

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

Momentum Conservation

- Inserting the propagator in this form introduces exponential factors for each end of the propagator.
- We can use the integral over z to integrate over the exponentials that are at the interaction term,

$$\int d^4z e^{-ip_1z} e^{-ip_2z} e^{-ip_3z} e^{ip_4z} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)$$

- This means that momentum is conserved at each interaction vertex.
- We are now left with a momentum integral instead at each vertex. Use these delta functions to perform as many of these integrals as possible.

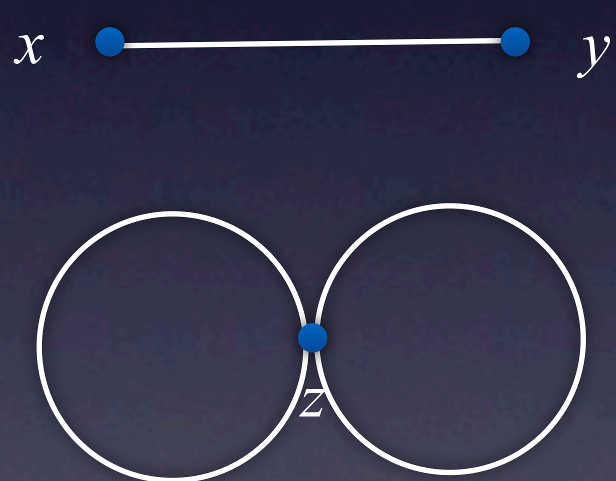
Feynman Rules

- The terms in the perturbative expansion can be written down as a diagrammatic set of rules. These are know as the *Feynman Rules*.
- For our simple example they are

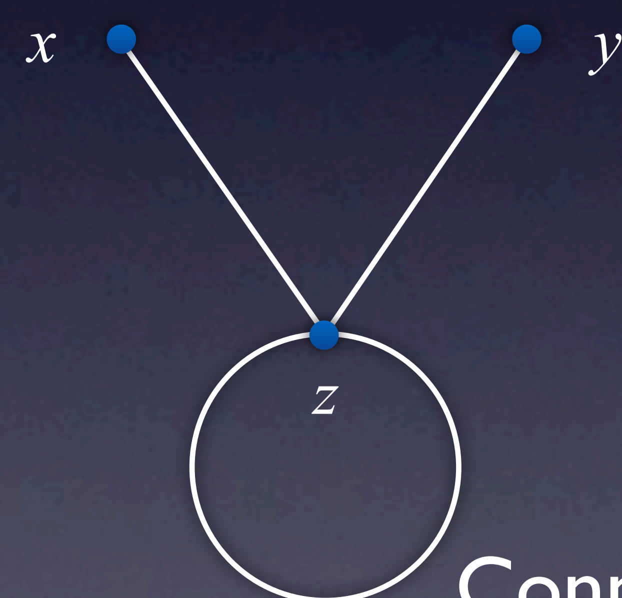
1. Write down all completely connected diagrams,
2. For each propagator write, $\frac{i}{p^2 - m^2 + i\epsilon}$
3. For each vertex write, $-i\lambda$
4. For each external point write, $e^{-ip\cdot x}$
5. Integrate over each undetermined momentum; $\int \frac{d^4p}{(2\pi)^4}$
6. Divide by the symmetry factor.

Disconnected Diagrams

- We apply these rules to produce only complete connected diagrams, i.e. every line must be connected to an external leg.



Disconnected



Connected

- Why can we drop the remaining disconnected diagrams?

Disconnected Diagrams

- The reason we drop them is due to the denominator $\langle 0|U(T,-T)|0\rangle$ present in,

$$\frac{\langle 0|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\exp\left[-i\int_T^{-T} dtH_I(t)\right]\}|0\rangle}{\langle 0|T\{\exp\left[-i\int_T^{-T} dtH_I(t)\right]\}|0\rangle}$$

- This denominator tells us to commute all vacuum diagrams as we compute U between two ground states.
- This factor is in the denominator and so will divide out all such states from the numerator.

Physical Processes

- We can now compute correlation functions using Feynman diagrams.
- How are these related to quantities we can measure in experiments?
- In particle physics experiments we typically want to compute (differential) cross sections or observables.
- For example the differential cross section,

$$d\sigma = \frac{|\mathcal{P}^2|}{\text{UnitTime} \times \text{UnitFlux}} d\Pi$$

Phase Space & Probability

$$d\sigma = \frac{|\mathcal{P}^2|}{\mathrm{UnitTime} \times \mathrm{UnitFlux}} d\Pi$$

- Where $d\Pi$ represents the phase space we must integrate over.
- This size of this phase space will depend upon the cuts imposed by both theoretical and experimental conditions.
- In these lectures we are learning how to compute $\mathcal{P}$, the probability of the particular process we are interested in occurring.
- The remaining parameters are determined by the set up of the computation.

Scattering

- We can think of a typical scattering process as consisting of three steps,
  - Start with a collection of particles which are *well separated* at some time in the distant past.
  - These particles then evolve through time and at some point can interact and possibly create new states.
  - The states then move apart and then in the far distant future they become *well separated* again.
- These will be our basic assumptions.

In States

- As this is a quantum process we cannot set up initial states of specific position and momentum instead we describe each initial state via a wave packet,

$$|\phi\rangle = \int \frac{d^3p}{(2\pi)^3} \phi(\vec{k})|\vec{k}\rangle \quad \langle\phi|\phi\rangle = 1 \quad \int \frac{d^3p}{(2\pi)^3} |\phi(k)|^2 = 1$$

- Here *Φ(k)* is the Fourier transform of the spatial wave-function.
- The *in* state is then constructed from a product of these wave-packets.
- Typically in a collider experiment there are two particles in the *in* state, which we will label A and B.

Out States

- After interacting the particles become well separated into n states in the distant future, the *out* states.
- The amplitude can be constructed by computing the overlap of the set of *in* states to the set of *out* states,

$$\mathcal{A} =_{\text{out}} \langle\phi_1\phi_2\ldots\phi_n|\phi_A\phi_B\rangle_{in}$$

- We have to make one assumption *that the asymptotic in and out states are the same as the free states of the theory.*

Asymptotic States

- Assuming that the asymptotic states are the same as the non-interacting free states allows us to proceed.
- It is not in fact true.
- Interacting field theories can never be truly separated at infinity so the asymptotic states are not the same as the free states.
- For example in QED we will see that there is a "cloud" of photons around electrons at infinity and similarly in QCD quarks will have a "cloud" of gluons.
- The effects of this assumption will manifest themselves in subtle ways as we will see later in these lectures.

The S-Matrix

- To perform computations using this amplitude we rewrite it as

$$\mathcal{A} =_{\text{out}} \langle \phi_1 \phi_2 \ldots \phi_n | \phi_A \phi_B \rangle_{in}$$
$$= \langle \phi_1 \phi_2 \ldots \phi_n | U(\infty, t_0) U(t_0, -\infty) | \phi_A \phi_B \rangle$$
$$= \langle \phi_1 \phi_2 \ldots \phi_n | S | \phi_A \phi_B \rangle$$

Assume the in and out states are free field states in the distant past and future

- We call S the scattering or S-Matrix of the theory.
- We can write it in terms of things we know how to compute as $S=U(\infty,-\infty)$.

Correlation Functions

- The *in* states at time, -T, can be written in terms of interaction picture field operators so that,

$$|\phi_A \phi_B\rangle = \phi(p_A)\phi(p_B)|0\rangle$$

- Similarly the out states at time, T, can be written as

$$\langle \phi_1 \phi_2 \ldots \phi_n| = \langle 0|\phi(p_1)\phi(p_2)\ldots\phi(p_n)$$

- We therefore want to compute,

$$\langle 0|\phi(p_1)\phi(p_2)\ldots\phi(p_n)S\phi(p_A)\phi(p_B)|0\rangle$$

$$= \langle 0|T\left\{\phi(p_1)\phi(p_2)\ldots\phi(p_n)\exp\left[-i\int_{-T}^{T} H_{\text{int}}(t)\right]\phi(p_A)\phi(p_B)\right\}|0\rangle$$

Time Limits

- We were able to move the *in* and *out* fields inside the Time ordering operation as they were defined at the beginning and end time of the computation.
- Our assumption is that these states are well separated at a time far in the past, this means that we must take the time limit to infinity.
- The amplitude can therefore be computed using Feynman diagrams as it is directly related to the correlation functions we investigated in the last lecture,

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\}|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\exp\left[-i\int_T^{-T} dt H_I(t)\right]\}|0\rangle}{\langle 0|T\{\exp\left[-i\int_T^{-T} dt H_I(t)\right]\}|0\rangle}$$

Take the limit in a slightly imaginary direction to ensure convergence of the exponentials.

The T-Matrix

- What we are really interested in are the effects of the interacting terms. This is the T-Matrix, T, which we define via,

$$S = 1 + iT$$

- If we compute T rather than S then we discard any terms where all the external legs are not connected to each other through some interaction.
- Overall momentum will be conserved, so if we extract a delta function factor from T to enforce this we can define the *matrix element* or *amplitude*,

$$\langle \phi_1 \phi_2 \ldots \phi_n | iT | \phi_A \phi_B \rangle = (2\pi)^4 \delta^{(4)} \left(p_A + p_B = \sum p_f \right) \cdot i\mathcal{A}(k_A, k_b \to p_f)$$

External Legs

- Examining our Feynman rules above we would associate with each external leg;
  - a factor of $\exp(-ip.x)$
  - a propagator connecting the interaction point to the external field
- As the external leg is on-shell though this propagator will diverge as,

$$\frac{i}{p^2 - m^2} \longrightarrow \frac{i}{0}$$

- These propagators though are related to the evolution of the external state up to the interaction point and do not describe any "scattering".

Amputated Legs

- We will therefore "amputate" this propagator from the external legs.
- We will also remove any "bubble loops" from this external leg, this will be important later.
- Effectively we are taking account that the external legs of the interacting theory are not the same as the free theory.
- This step can be put on a much more formal footing through the use of the LSZ reduction formula.
- The remaining exponentials will then form part of the overall momentum conserving delta function we extracted in defining the T-Matrix, so we will also drop these pieces.

Feynman Rules

- Our final procedure for computing an amplitude is given by writing down all *amputated, fully connected* Feynman diagrams, with the following final Feynman rules;

1. For each propagator, $\frac{i}{p^2 - m^2 + i\epsilon}$
2. For each vertex, $-i\lambda$
3. For each external line, 1
4. Impose momentum conservation at each vertex.
5. Integrate over each undetermined momentum; $\int \frac{d^4p}{(2\pi)^4}$
6. Divide by the symmetry factor.

Spinor & Vector Propagators

- These are the Feynman rules for a scalar field theory for other theories that we will want to consider we will have to alter rules *1* and 2.
- For a spinor propagator we write,

$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

- For a photon propagator we would have

$$\frac{ig^{\mu\nu}}{p^2-m^2+i\epsilon}$$

Cross sections

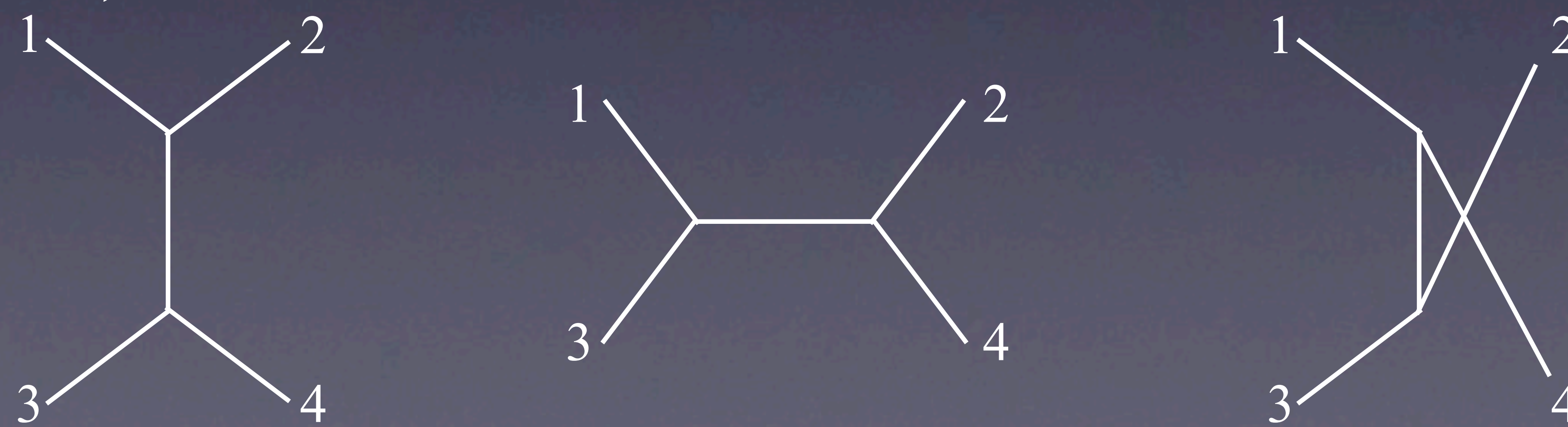
- After computing A we can then compute the cross section (or more useful a differential cross section).
- For two incoming particles this is given by,

$$d\sigma = \frac{1}{4E_A E_B}\frac{1}{|\vec{v}_A - \vec{v}_B|}|\mathcal{A}(k_A, k_b \to p_f)|^2 d\Pi$$

- Depending upon the quantity we are interested in we can perform some or all of the integrals $d\Pi$. This is know as integrating over the phase space.

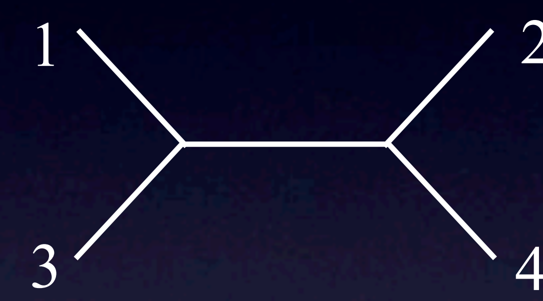
Example 1

- To see how this works in practice let us look at a couple of examples.
- Start with a simple example of a lowest order, called tree level, perturbative computation in a Φ^3 scalar theory.
- Here the Feynman rule for the interaction is simply $-i\lambda$, where again λ is the coupling constant.
- There are three diagrams for the process $\Phi_1\Phi_2 \to \Phi_3\Phi_4$ they are,



Example 1

- Let us examine two of these Feynman diagrams,



- Use the Feynman rules to get,

$$(-i\lambda)^2 \int \frac{d^4p}{(2\pi)^4}\Bigg(\delta^{(4)}(p_1+p_2-p)\delta^{(4)}(p-p_3-p_4)\frac{i}{p^2-m^2+i\epsilon}$$

$$+\delta^{(4)}(p_1-p_3-p)\delta^{(4)}(p-p_2+p_4)\frac{1}{p^2-m^2+i\epsilon}\Bigg)$$

Example 1

- We can use one of the delta functions to perform the integral

$$\int \frac{d^4p}{(2\pi)^4}\delta^{(4)}(p_1 - p_3 - p)\delta^{(4)}(p - p_2 + p_4)\frac{1}{p^2 - m^2 + i\epsilon}$$
$$= \frac{\delta^{(4)}(p_1 + p_2 - p_3 - p_4)}{(p_1 - p_3)^2 - m^2 + i\epsilon}$$

- This gives us the more compact expression,

$$i(-i\lambda)^2\left[\frac{1}{(p_1 + p_2)^2 - m^2} + \frac{1}{(p_1 - p_3)^2 - m^2}\right]$$

- After adding the third diagram to this we have the full amplitude that can then be used to compute the cross section.

Example II

- As a second example consider the next to lowest perturbative expansion term, this is usually called the one-loop term, for $\Phi_1 \rightarrow \Phi_2$

- This gives the amplitude

$$i^2(-i\lambda)^2 \int \frac{d^4p}{(2\pi)^2} \frac{1}{p^2 - m^2} \frac{1}{(p_1 - p)^2 - m^2}$$

- This expression retains an integration over an internal momentum which needs to be performed.

Summary

- We now know how to write down Feynman Diagrams.
- We have defined the *In* and *Out* states and their relation to the S-Matrix.
- We showed how we can connect these correlation functions to the computation of cross sections.

Lecture 6

- How do we constructing a Lagrangian?
  - Generically what type of terms can we can incorporate?
- Investigate the fundamental role different types of symmetries play in the construction of field theories such as QED and QCD.
  - Global,
  - Local,
  - Gauge.

Symmetries

- We are now ready to look at more complicated gauge field theories such as QED and QCD.
- One key feature of modern physics is the use of symmetries.
- In gauge field theories symmetries play an important role in determining the structure of the Lagrangian.
- We will see that by demanding the Lagrangian is invariant under certain symmetries we are forced to introduce interaction terms of the type we desire.

Perturbations

- What additional terms can we add to a Lagrangian which also satisfy our demands that we can compute using a perturbative expansion?
- In our example cases we assumed the coupling constant λ was very small.
- In general we cannot quite use this argument to limit the terms we want in our Lagrangians.
- Fields carry a "mass dimension".

Power Counting

- Consider a Lagrangian

$$L = \int d^4x \mathcal{L} \qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n$$

- The Lagrangian, L, carries "dimension", [L]=0 and as $[d^4x]$=-4 then the Lagrange Density has mass dimension

$$[\mathcal{L}] = 4$$

- The other terms in the Lagrangian carry mass dimensions

$$[\phi] = 1, \quad [m] = 1, \quad [\lambda_n] = 4 - n$$

Relevant Terms

- We need to take into account the dimension of the coupling of a term to determine if it is “small”;
- $[\lambda_3] = 1$, For this term, the dimensionless parameter is λ_3/E, where E has dimensions of mass. Typically in quantum field theories E is the energy scale of the process of interest.
- This means that $\lambda_3\ \Phi^3/3!$ is a small perturbation at high energies $E \gg \lambda_3$, but a large perturbation at low energies $E \ll \lambda_3$.
- Terms that we add to the Lagrangian with this behaviour are called *relevant* because they have the most impact at low energies (which is where most of the physics we see lies).

Marginal & Irrelevant Terms

- The other types of term fall into two categories,
  - $[\lambda_4] = 0$, this term is small if $\lambda_4 \ll 1$. Such terms are called marginal.
  - $[\lambda>4] < 0$, for $n \geq 5$ the dimensionless parameter is $\lambda_n E^{n-4}$, which is small at low-energies and large at high energies. Such terms are called irrelevant.
- Irrelevant terms lead to "non-renormalisable" field theories, where one cannot make sense of the infinities at arbitrarily high energies.
- Irrelevant terms are not necessarily to be avoided, they just signify that the theory is incomplete in the high energy regime.

Reduced Lagrangian

- We will restrict ourselves to Lagrangians that contain only relevant and marginal terms as we want to deal with renormalisable theories.
- Also the physics we are interested in occurs far below some "GUT" scale, so irrelevant operators will be greatly suppressed.
- Irrelevant operators are very important in Effective Field Theory.
- This drastically reduces the number of possible terms we can have in our Lagrangian.

Global Symmetries

- The Lagrangian typically contains a number of *Global symmetries*.
- A global symmetry is a transformation that acts only on the fields of the theory and is the same at every point in space-time.
- For example a spinor field is invariant under transformations of the type,

$$\Psi \rightarrow e^{-i\alpha}\Psi \qquad \overline{\Psi}\gamma^{\mu}\Psi \rightarrow \overline{\Psi}e^{i\alpha}\gamma^{\mu}e^{-i\alpha}\Psi = \overline{\Psi}\gamma^{\mu}\Psi$$

- So that the Lagrangian remains invariant under this global *U(1)* symmetry.

Lie Algebras

- The symmetry groups we will look at are all Lie Algebras.
- Of all the Lie Algebras we will only use two of them in these lectures, *U(N)* and *SU(N)*.
- *U(N)* is the group of all unitary $N \times N$ matrices.
- So *U(1)* is just a constant.
- *SU(N)* is the group of all unitary $N \times N$ matrices with determinant equal to 1. We will see more of this in the next lecture.

Global Symmetries

- Global symmetries of this kind lead to conserved currents.
- Noethers Theorem tells us that every continuous symmetry gives rise to a conserved current, $j(x)$, such that the equation of motion is given by,

$$\partial_\mu j^\mu = 0$$

- For each conserved current there is a corresponding conserved charge,

$$Q = \int_{R^3} d^3x j^0$$

Local Symmetries

- A more interesting class of symmetries arrises when we consider making the symmetry depend upon space-time.
- As an example consider making the global *U(1)* transformation a local *U(1)* transformation,

$$\Psi \to e^{-i\alpha(x)}\Psi$$

- If we insert this into the Dirac Lagrangian then we get,

$$\overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi \to \overline{\Psi}e^{i\alpha(x)}\left(i\partial\!\!\!/ - m\right)e^{-i\alpha(x)}\Psi$$

Covariant Derivative

$$\overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi \rightarrow \overline{\Psi}e^{i\alpha(x)}\left(i\partial\!\!\!/ - m\right)e^{-i\alpha(x)}\Psi$$

- Unlike in the global case the differential will now act on the exponential as it depends on x,

$$\overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi + \overline{\Psi}e^{i\alpha(x)}e^{-i\alpha(x)}(\partial\!\!\!/\alpha(x))\Psi$$

- So this Lagrangian is not invariant under this local transformation.
- To make this invariant we need to add an additional term which will make it invariant.
- To achieve this first replace the standard derivative, ∂ , with the covariant derivative D,

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + ieA_\mu$$

Covariant Derivative

- Finally the Lagrangian is invariant if A_μ transforms simultaneously under the gauge transformation,

$$A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$$

- The complete transform of the covariant derivative is then just a change of phase,

$$\begin{aligned} D_\mu \Psi &\rightarrow \partial_\mu \left(e^{-i\alpha(x)}\Psi\right) + i\left(A_\mu + \partial_\mu \alpha(x)\right)\left(e^{-i\alpha(x)}\Psi\right) \\ &= e^{-i\alpha(x)} D_\mu \Psi \end{aligned}$$

- This invariant transformation is know as a local gauge transformation.

A New Interaction

- By requiring a local gauge transformation we see that we have had to introduce an additional term into the Lagrangian,

$$-e\overline{\Psi}A_\mu\Psi$$

- This is the coupling of two spinors to a vector field with a coupling constant e.
- By demanding a local invariance on a symmetry we have been constrained to introduce interactions!
- This idea of imposing a symmetry to introduce interaction terms is the basis for both QED, EW, QCD and many other models.

Photons

$$-e\overline{\Psi}A_\mu\Psi$$

- We have one problem though, we have an interaction term for a field A_μ but we have no kinetic term.
- We would like to associate the vector field with the Electro-Magnetic (EM) field, with e the coupling constant of the photon to the electron.
- We must therefore also add the kinetic term, the only term that we can add turns out to be the classical Lagrangian for Maxwell's equations,

$$\mathcal{L}_{EM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- With the Field Strength tensor given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Maxwells Equations

- The classical Lagrangian for Maxwell's equations,

$$\mathcal{L}_{EM} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- This Field Strength tensor satisfies the Bianchi identity,

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

- Using this and the equations of motion of the field we can derive all of Maxwells equations (in the absence of charged matter),

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\nabla \cdot \vec{B} = 0 \qquad \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\nabla \cdot \vec{E} = 0 \qquad \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B}$$

Gauge Symmetries

- After adding this term we can now see why we described,

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$$

- as a local gauge transformation.
- This is simply the gauge transformation that Maxwells equations are invariant under,

$$F_{\mu\nu} \to \partial_\mu \left(A_\nu - \frac{1}{e}\partial_\nu \alpha(x) \right) - \partial_\nu \left(A_\mu - \frac{1}{e}\partial_\mu \alpha(x) \right) = F_{\mu\nu}$$

- Unlike a global transformation which simply takes a physical state to another physical state with the same properties a gauge transformation represents a redundancy in the description of the system.
- We identify a state A_μ as being the same state as $A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$

Gauge Choice

- This gauge freedom means that an additional constraint or gauge choice is required in order to quantise the theory.
- Two possible choices we could make are,
  - The Lorenz gauge where $\partial_\mu A^\mu = 0$
  - The Coulomb gauge where $\vec{\nabla} \cdot \vec{A} = 0$
- We will use the Lorenz gauge as it preserves Lorentz invariance.

The QED Lagrangian

- If we assemble all the pieces together then we have the Lagrangian for QED,

$$\mathcal{L}_{\mathrm{QED}} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi - e\Psi\gamma^{\mu}A_{\mu}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

- This is amazingly simple considering the amount of physics it describes!
- It consists of a photon and an electron whose coupling to each other is simply the electric charge.

QED Gauge Choice

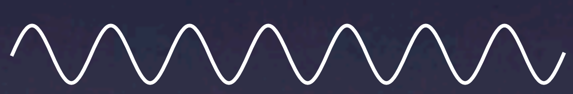
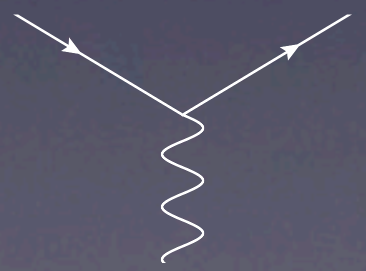
- When quantising the theory we will want to add an additional term to the Lagrangian to take into account our gauge fixing choice,

$$\mathcal{L}_{\mathrm{QED}} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi - e\Psi\gamma^{\mu}A_{\mu}\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2\alpha}(\partial^{\mu}A_{\mu})^2$$

- We can choose different values for α, we generally also call these different values "gauge choices"
  - Feynman Gauge: $\alpha=1$
  - Landau Gauge: $\alpha=0$ (set in the Feynman rules)

QED Feynman Rules

$$\mathcal{L}_{\mathrm{QED}} = \overline{\Psi}\left(i\partial\!\!\!/ - m\right)\Psi - e\Psi\gamma^{\mu}A_{\mu}\Psi - \frac{1}{2}F^{\mu\nu}F_{\mu\nu}$$

- From this Lagrangian we have the following Feynman Rules,
  - We denote every photon propagator using 
  - We denote each electron propagator using 
  - For each vertex we have 

QED Feynman Rules

- For each external photon we have a factor of ϵ^{μ}.
- For each external electron we have a factor of $\overline{u},\ u,\ \overline{v},\ v.$
- We also have the following rules,

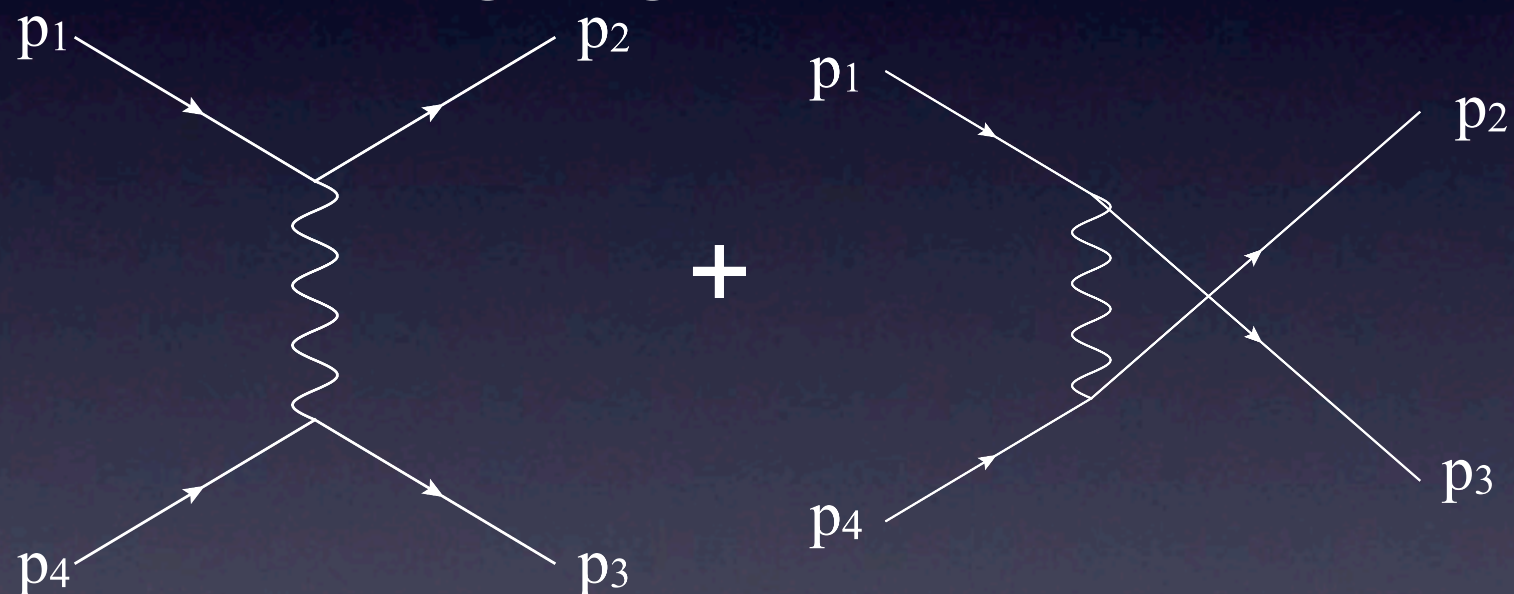
$$= \frac{-i\left(g^{\mu\nu} + (\alpha - 1)\frac{p^{\mu}p^{\nu}}{p^2}\right)}{p^2 + i\epsilon}$$

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

$$= -ie\gamma^{\mu}$$

QED Examples I

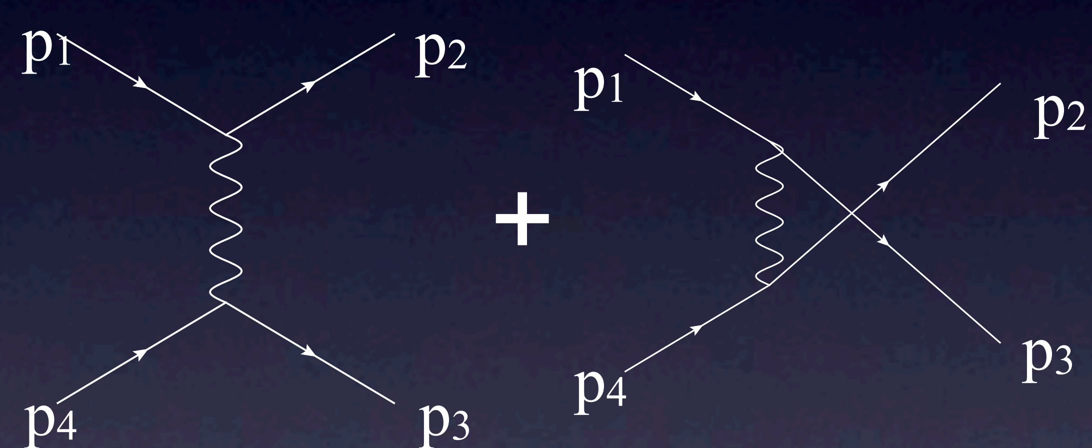
- Electron Scattering: $e^- e^- \to e^- e^-$
- Two contributing diagrams



$$= -i(-ie)^2 \left(\frac{[\overline{u}(p_1)\gamma^\mu u(p_2)][\overline{u}(p_3)\gamma_\mu u(p_4)]}{(p_2 - p_1)^2} - \frac{[\overline{u}(p_1)\gamma^\mu u(p_3)][\overline{u}(p_2)\gamma_\mu u(p_4)]}{(p_1 - p_3)^2} \right)$$

QED Gauge Issues

- Look at this again but try it in a different gauge.



$$= -i(-ie)^2\left(\frac{[\overline{u}(p_1)\gamma^\mu u(p_2)][\overline{u}(p_3)\gamma^\nu u(p_4)]}{(p_2-p_1)^2}\left(g_{\mu\nu}+(\alpha-1)\frac{(p_2-p_1)^\mu(p_2-p_1)^\nu}{(p_2-p_1)^2}\right)\right.$$

$$\left.-\frac{[\overline{u}(p_1)\gamma^\mu u(p_3)][\overline{u}(p_2)\gamma^\nu u(p_4)]}{(p_1-p_3)^2}\left(g_{\mu\nu}+(\alpha-1)\frac{(p_1-p_3)^\mu(p_1-p_3)^\nu}{(p_1-p_3)^2}\right)\right)$$

Amplitudes

- Two of the terms are the same as before,

$$= -i(-ie)^2 \left(\frac{[\overline{u}(p_1)\gamma^\mu u(p_2)][\overline{u}(p_3)\gamma_\mu u(p_4)]}{(p_2 - p_1)^2} - \frac{[\overline{u}(p_1)\gamma^\mu u(p_3)][\overline{u}(p_2)\gamma_\mu u(p_4)]}{(p_1 - p_3)^2} \right)$$

- but we have two additional terms,

$$= -i(-ie)^2(\alpha - 1) \left(\frac{[\overline{u}(p_1)(\not{p}_2 - \not{p}_1)u(p_2)][\overline{u}(p_3)(\not{p}_2 - \not{p}_1)u(p_4)]}{((p_2 - p_1)^2)^2} - \frac{[\overline{u}(p_1)(\not{p}_1 - \not{p}_3))u(p_3)][\overline{u}(p_2)(\not{p}_1 - \not{p}_3)u(p_4)]}{((p_1 - p_3)^2)^2} \right)$$

- These terms go to zero and so we get the same result as before.

- In general the amplitude is gauge invariant though individual Feynman diagrams are not.

Amplitudes

- Two of the terms are the same as before,

$$= -i(-ie)^2\left(\frac{[\overline{u}(p_1)\gamma^\mu u(p_2)][\overline{u}(p_3)\gamma_\mu u(p_4)]}{(p_2-p_1)^2} - \frac{[\overline{u}(p_1)\gamma^\mu u(p_3)][\overline{u}(p_2)\gamma_\mu u(p_4)]}{(p_1-p_3)^2}\right)$$

- but we have two additional terms,

$$= -i(-ie)^2(\alpha-1)\left(\frac{[\overline{u}(p_1)(\not{p}_2-\not{p}_1)u(p_2)][\overline{u}(p_3)(\not{p}_2-\not{p}_1)u(p_4)]}{((p_2-p_1)^2)^2} - \frac{[\overline{u}(p_1)(\not{p}_1-\not{p}_3))u(p_3)][\overline{u}(p_2)(\not{p}_1-\not{p}_3)u(p_4)]}{((p_1-p_3)^2)^2}\right)$$

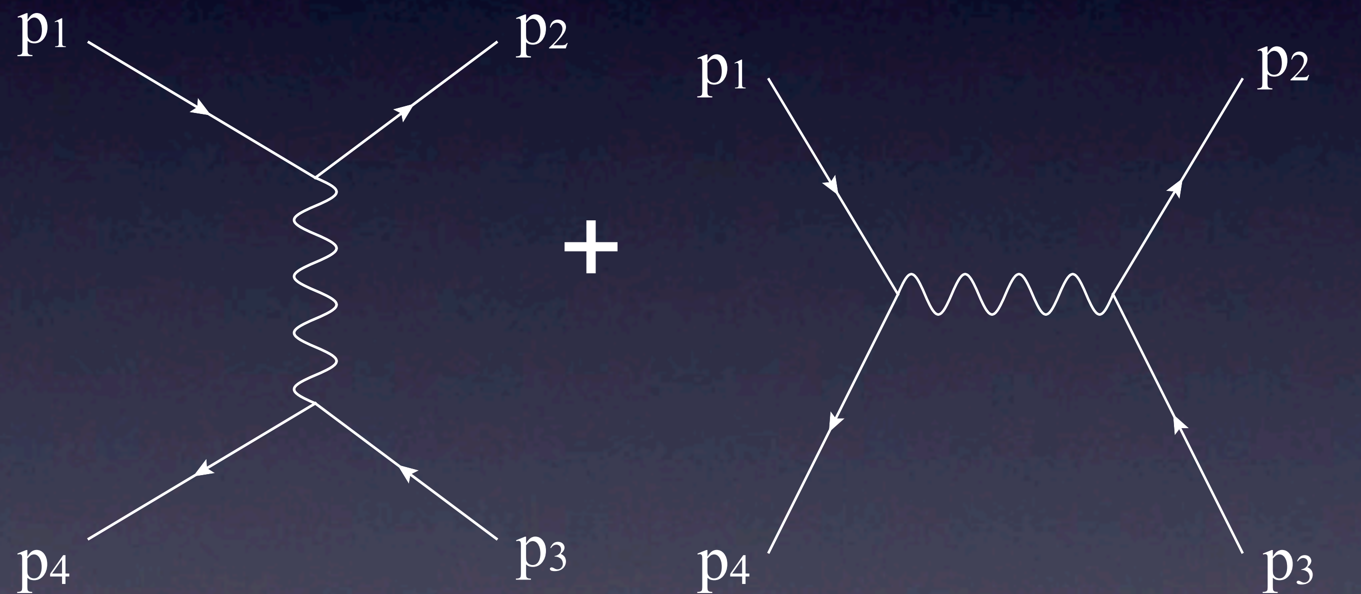
Vanish

- These terms go to zero and so we get the same result as before.

- In general the amplitude is gauge invariant though individual Feynman diagrams are not.

QED Examples II

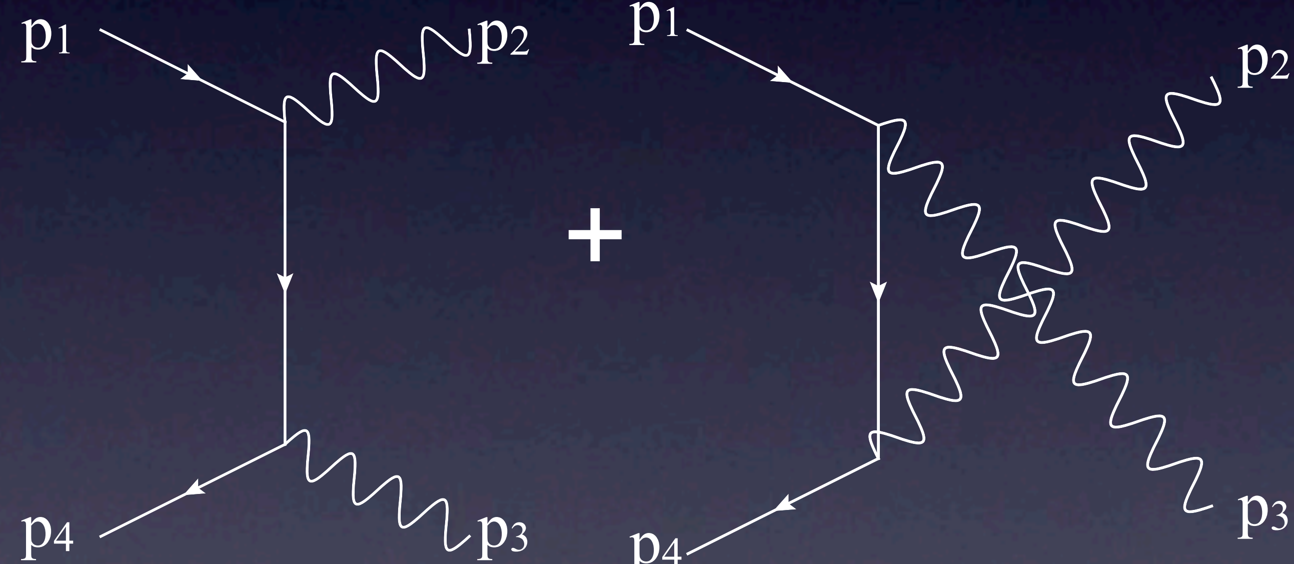
- Electron Positron Scattering : $e^-e^+ \to e^-e^+$
- Two contributing diagrams



$$= -i(-ie)^2 \left(\frac{[\overline{u}(p_1)\gamma^\mu u(p_2)][\overline{v}(p_4)\gamma_\mu v(p_3)]}{(p_2 - p_1)^2} - \frac{[\overline{u}(p_1)\gamma^\mu u(p_4)][\overline{v}(p_2)\gamma_\mu v(p_3)]}{(p_1 + p_2)^2} \right)$$

QED Examples III

- Electron Positron Annihilation : $e^-e^+ \rightarrow \gamma\gamma$
- Two contributing diagrams



$$= i(-ie)^2 \overline{v}(p_4) \left(\frac{\not\epsilon(p_3)(\not p_1 - \not p_2 + m)\not\epsilon(p_2)}{(p_1 - p_2)^2 - m^2} + \frac{\not\epsilon(p_2)(\not p_1 - \not p_3 + m)\not\epsilon(p_3)}{(p_1 - p_3)^2 - m^2} \right) u(p_1)$$

Mandelstam Variables

- When computing $2 \rightarrow 2$ Feynman Diagrams we will come across similar combinations of the external momenta repeatedly.
- These standard combinations are known as the Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2$$
$$t = (p_1 - p_1')^2 = (p_2 - p_2')^2$$
$$u = (p_1 - p_2')^2 = (p_2 - p_1')^2$$

- Here p_1 and p_2 are the momenta of the two initial particles, and p'_1 and p'_2 are the momenta of the final two particles.
- When each momenta has mass $M^2{}_i$ these variables satisfy,

$$s + t + u = \sum_i M_i^2$$

Summary

- We have seen what type of terms we can incorporate into a Lagrangian.
- Investigate the fundamental role different types of symmetries played in the construction of field theories such as QED and QCD.
  - Global,
  - Local,
  - Gauge.