Introduction to Quantum Field Theory and QCD

Lecture 9 & 10

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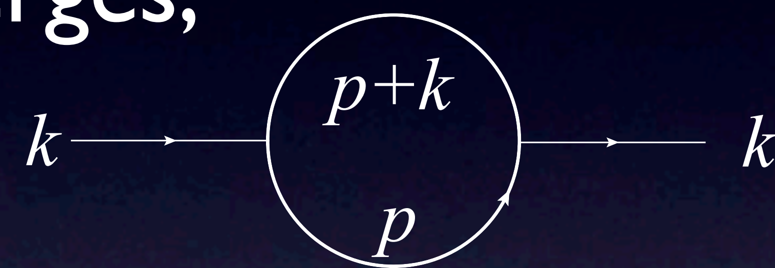
Lecture 9

- We will finish our investigation into Renormalisation.
- Look at computing Next-to-Leading Order (NLO) Corrections.
- Understand IR singularities.

One-loop integrals

- In the last lecture we saw that the one-loop integral diverges,



- To deal with this situation we will regulate the integral using Dimensional Regularisation,

$$C\int_a^\infty d|\vec{p}||\vec{p}|^{-1-\epsilon} = \frac{1}{\epsilon}a^{-\epsilon} = \frac{1}{\epsilon}e^{-\epsilon\ln a} = \frac{1}{\epsilon} - \ln a + \ldots$$

- So we have poles in ε which we want to remove.

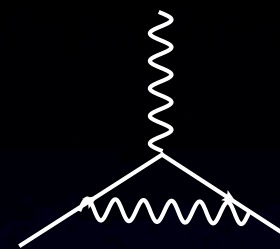
- To do this we must renormalise our theory.

QED & QCD Loop Corrections

- To see how renormalisation works let us consider a more complicated example than the bubble.
- Look at the QED/QCD *Vertex correction*.
- The basic vertex looks like,

- The one-loop corrections look like,

Vertex Correction



- We get the following expression for this,

$$\int \frac{d^D l}{(2\pi)^D} \gamma^\alpha \frac{i(\not{l} + \not{p}_2 + m)}{(l+p_2)^2 - m^2 + i\epsilon} (-ie\gamma^\mu) \frac{i(\not{l} + \not{p}_1 + m)}{(l+p_1)^2 - m^2 + i\epsilon} \frac{1}{l^2 + i\epsilon}$$

- There are similar results for the other three terms.

- The sum of the terms after integration will have the following structure,

$$= eC_1 + \left(\frac{a}{\epsilon} + b\right) e^3$$

Renormalising the Vertex Correction

- Again we have an unwanted ε in our result.
- To remove this we will renormalise.
- What is renormalisation?
- The parameters in the Lagrangian, such as the coupling constants and masses, are not the actual parameters we measure in an experiment.
- To renormalise we relate the bare parameters of the Lagrangian to the actual measurable quantities.
- We effectively absorb the divergent pieces into a redefinition of the parameters.

Coupling Constant

- For the vertex correction we need to renormalise the electromagnetic coupling constant.

$$e = Z_e e_R$$

- QED is a renormalisable theory so we only need a finite number of renormalisable parameters.
- We can compute the renormalisation parameters order by order in perturbation theory.
- To proceed therefore we will compute our perturbative expansion as before in terms of bare parameters.
- Then replace the bare parameters with the redefinition above.

The Coupling Constant

- Let us see how this will work for the charge renormalisation term,

$$e = Z_e e_R$$

- We can write the Z_e as a perturbative expansion in terms of our dimensionally regularised result,

$$Z_e = \left(1 + \frac{Z_e^{(1)}}{\epsilon} e_R^2 + \left(\frac{Z_e^{(2)}}{\epsilon^2} + \frac{Z_e^{(1)}}{\epsilon}\right) e_R^4 + \dots\right)$$

- This can then be inserted into our perturbative expression in terms of the bare parameters,

$$eC_1 + \left(\frac{a}{\epsilon} + b\right) e^3$$

- After dropping terms higher order in e_R we have,

$$\left(1 + \frac{Z_e^{(1)}}{\epsilon} e_R^2 + \dots\right) e_R C_1 + \left(\frac{a}{\epsilon} + b\right) (1 + \dots) e_R^3$$

Coupling Constant

$$\left(1+\frac{Z_e^{(1)}}{\epsilon}e_R^2+\dots\right)e_RC_1+\left(\frac{a}{\epsilon}+b\right)(1+\dots)e_R^3$$

- We can now choose $Z_e^{(1)}$ such that we cancel the pole terms,

$$C_1\frac{Z_e^{(1)}}{\epsilon}e_R^3+\frac{a}{\epsilon}e_R^3=0$$

- Leads to the expression,

$$Z_e^{(1)}=-\frac{a}{C_1}$$

- The renormalised result is now finite and given by,

$$e_R+be_R^3$$

- This renormalised electric charge is the physical charge we measure, all the divergent terms have been absorbed into it.

Counter Terms

- Rather than computing our expressions in terms of the bare parameters it is usually more efficient to work with a Lagrangian written directly in terms of the renormalised fields and parameters.
- Rewrite the Lagrangian in terms of the renormalised parameters at the expense of adding additional UV counter-terms to the Lagrangian to compensate for this.
- It can be shown then that for each renormalisation parameter we add an additional term to the Lagrangian.
- We can compute this contribution in a perturbation series, e.g. in QED we would add the new vertex,

$$-e_R(Z_e - 1)\overline{\Psi}\gamma^\mu\Psi A_\mu$$

Counter Terms

$$-e_R(Z_e - 1)\overline{\Psi}\gamma^\mu \Psi A_\mu$$

- With this new Lagrangian the computation that we had before would then become,

$$= e_R C_1 + \left(\frac{a}{\epsilon} + b\right) e_R^3 - e_R(Z_e - 1)$$

- The Z_e will be exactly as before,

$$Z_e = \left(1 + \frac{Z_e^{(1)}}{\epsilon} e_R^2 + \left(\frac{Z_e^{(2)}}{\epsilon^2} + \frac{Z_e^{(1)}}{\epsilon}\right) e_R^4 + \ldots\right)$$

- Again we choose the parameters to cancel the poles, so that,

$$Z_e^{(1)} = -\frac{a}{C_1}$$

- Again we have a finite result.

Coupling Constant

- On the surface this procedure might seem somewhat ad-hoc.
- There seems to be a lot of freedom in our choice for the coefficients of these renormalisation terms, but there is a limit to the number of terms we can fix in this way.
- We choose coefficients such that they cancel the UV poles.
- This is a self consistent approach. Once we have chosen the coefficient to remove one type of divergence we cannot change it again to remove another divergence elsewhere.
- The choice once made is universal and works to remove *all* UV divergent terms in the computation.
- This consistent choice is known as a *renormalisation scheme*.

Finite Results

- A similar procedure applied to all the other bare parameters in the theory leaves us with a finite result up to a particular order in the perturbation series.
- We have removed the one-loop divergence in $(e_R)^3$, but not at higher orders in e_R.
- A consequence of this perturbative renormalisation is that we introduce a renormalisation scale, μ_R.
- This unphysical scale would drop out of any full result, but we will be left with a higher order dependance in a perturbative computation.
- This leads to the identity,

$$\mu_R \frac{de_R(\mu_R)}{d\mu_R} = \beta(e_R(\mu_R))$$

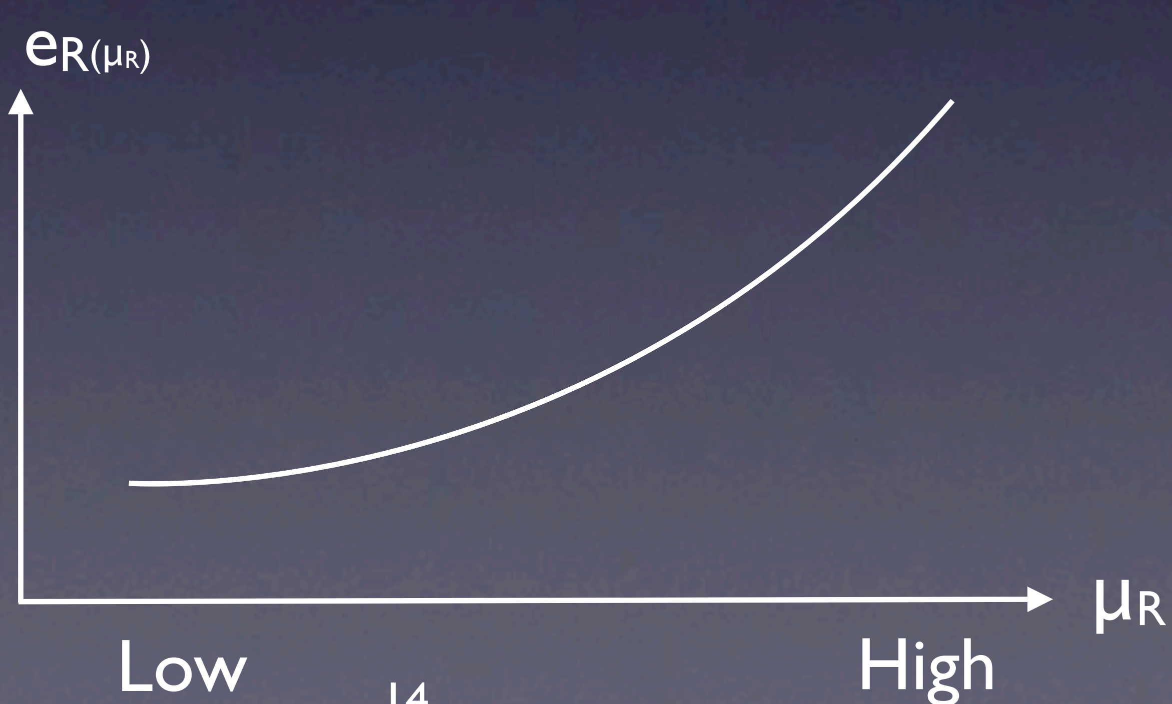
The Beta Function

$$\mu_R \frac{de_R(\mu_R)}{d\mu_R} = \beta(e_R(\mu_R))$$

- This beta function tells us how the coupling constant evolves with a change of scale.
- It is computed in a perturbative expansion in terms of the coupling,

$$\beta(e_R(\mu_R)) = \beta_1 e_R^3(\mu_R) + \beta_2 e_R^5(\mu_R) + \ldots$$

- For QED this leads to the renormalisation scale dependance,



The QCD Beta Function

- We can perform a similar computation in QCD but this time the beta function has a minus sign in front of it.

$$\beta_0 = 11 - \frac{2}{3} n_f$$

$$\mu_R \frac{d\alpha_S(\mu_R)}{d\mu_R} = -\frac{\beta_0}{4\pi}\alpha_S^2(\mu_R) - \frac{\beta_1}{(4\pi)^2}\alpha_S^3(\mu_R) - \ldots$$

- This leads to the famous asymptotic freedom,

NLO Computations

- We can now compute tree level and one-loop level amplitudes
- Combine these together to derive the next-to-leading order (NLO) contribution to a perturbative series.
- This will not be as straightforward as it would first appear,
  - Collinear and Infra-red (IR) divergences will cause problems.

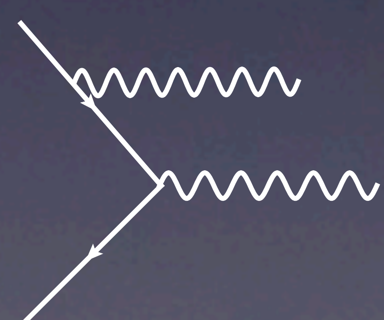
NLO Contributions

- The perturbative expansion consists of,

LO:  $= A_0$

 $+ \ldots = g^2 A_2$

NLO:

 $+ \ldots = g A_1$

$$A_{NLO} = A_0 + gA_1 + g^2 A_2$$

Squared Amplitudes

$$A_{NLO} = A_0 + gA_1 + g^2 A_2$$

- Squaring this amplitude to produce a cross section or observable shows us why we must include both the real and virtual terms,

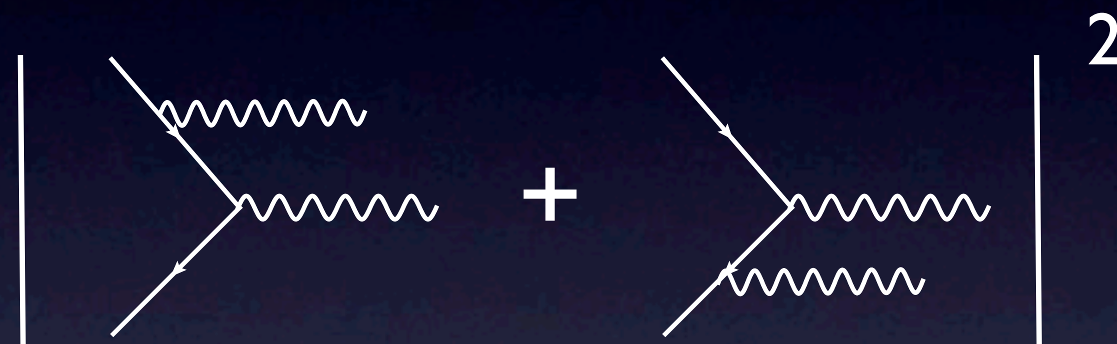
$$|A_{NLO}|^2 = |A_0|^2 + g^2|A_1|^2 + g^2(A_2^* A_0 + A_0^* A_2)$$

Real Virtual

- Unlike for the LO terms and the real pieces the virtual piece can be negative.
- The NLO term can therefore also be negative.

Real Diagrams

- For the real contribution we sum and then square (as this is QM)

$|A_1|^2 =$ 

- The phase space integral is now more complicated as it is over two particles.

$$\sigma_R^{(1)} = \frac{1}{2s}\frac{1}{4N}\int d\Pi_2(q,k)\sum_s |A_1|^2$$

Real Phase Space

- The two particle phase space integral is given by,

$$\int d\Pi_2 = \int \frac{d^3k}{(2\pi)^3 2k^0} \frac{d^3q}{(2\pi)^3 2q^0}$$

- Examine "half" of this,

$$\int \frac{d^3k}{(2\pi)^3 2k^0} = \int \frac{k^2 dk d\cos\Theta d\psi}{(2\pi)^3 2k^0}$$

- After summing and squaring the amplitude we get at least one term of the type,

$$\sum |A_1|^2 = \alpha_S^2 e \frac{2(p_1 \cdot p_2)}{(p_1 \cdot k)(p_2 \cdot k)} + \text{others}$$

In More Detail

- Examine this term in more detail by choosing a particular momentum parameterisation,

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)$$
$$p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)$$
$$k = |\vec{k}|(1,0,\sin\Theta,\cos\Theta)$$

- So that the amplitude squared becomes,

$$\sum |A_1|^2 = \frac{s}{\frac{s}{4}(1-\cos\Theta)(1+\cos\Theta)k^2}$$

- The part of the phase space we are interested in is then given by

$$\int d\Pi_2 \sum |A_1|^2 = \alpha_S C \int \frac{dk^0}{k^0} \int_{-1}^{1} \frac{d\cos\Theta}{1-\cos^2\Theta}$$

IR Divergences

- Examining this expression we see that there are two sources of divergence,

$$\int d\Pi_2 \sum |A_1|^2 = \alpha_S C \int \frac{dk^0}{k^0} \int_{-1}^{1} \frac{d\cos\Theta}{1-\cos^2\Theta}$$

Infra-red Divergence

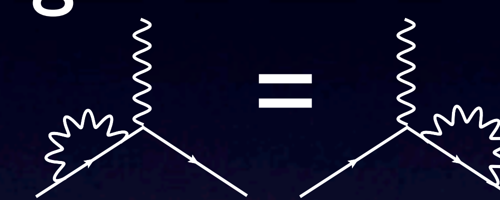
Collinear Divergence

$$\int_0^\pi \frac{d\Theta}{\sin\Theta} \approx \int_0^\pi \frac{d\Theta}{\Theta}$$

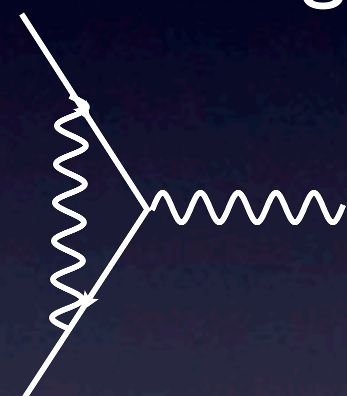
- So there are two sources of divergence.
- How do we deal with these, we cannot remove them in the same way as UV divergences.

Virtual Diagrams

- The virtual amplitude contribution will also contain poles that we can regulate using Dimensional Regularisation,



$$= 0$$



$$\propto \alpha_S \left(\frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C \right)$$

Both IR and collinear Divergences

- The cross section contribution will then be,

$$\sigma_V^{(1)} = \frac{1}{2s} \frac{1}{4N} \int d\Pi_1 \sum_s (A_2^* A_0 + A_0^* A_2)$$

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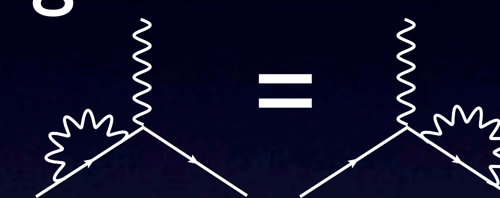

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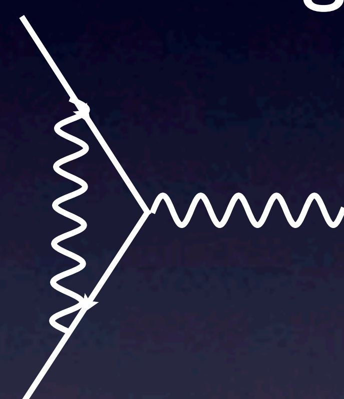

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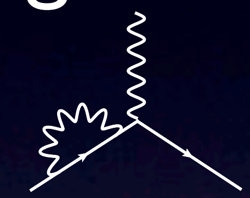
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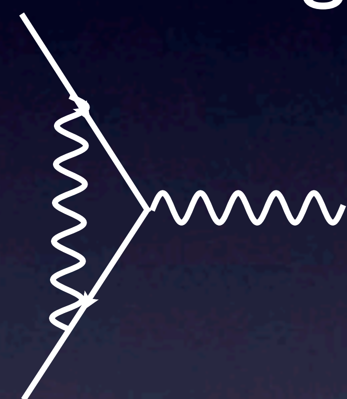
$= 0$



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$$= \frac{2\pi}{s}\delta\left(1 - \frac{k^2}{s}\right)$$

$$\propto \alpha_S \left(\frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C \right) \delta(1-z)$$

Cancelling Divergences

- The IR divergence's simply cancel with divergences in the virtual part,

$$\sigma_V^{(1)} \approx \alpha_S \frac{-A}{\epsilon^2} \delta(1-z)$$

$$\sigma_R^{(1)} \approx \alpha_S \frac{A}{\epsilon} (1-z)^{-1+\epsilon}$$

$$\alpha_S \frac{A}{\epsilon} \left(\frac{1}{\epsilon} \delta(1-z) + \frac{1}{[1+z]_+} + \epsilon \left(\frac{\ln(1-z)}{(1-z)} \right)_+ \right)$$

- The plus distribution is defined as,

$$\int_0^1 dz \frac{f(z)}{1-z}_+ = \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$

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Cancel

$$\alpha_S \frac{A}{\epsilon}\left(\frac{1}{\epsilon}\delta(1-z) + \frac{1}{[1+z]_+} + \epsilon\left(\frac{\ln(1-z)}{(1-z)}\right)_+\right)$$

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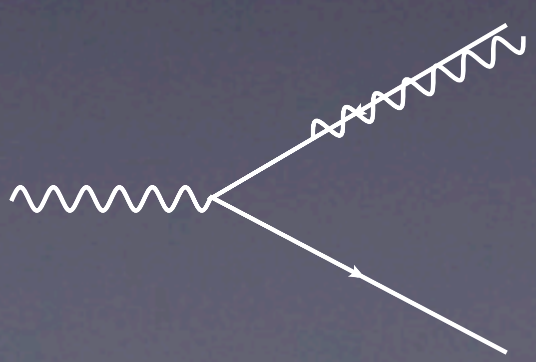
Bloch-Nordsieck

- The Bloch-Nordsieck theorem tells us that IR divergences will always cancel between the real and virtual terms.
- This differs from the UV divergences that we had to remove using renormalisation.
- What about the Collinear divergences?
- To deal with these we will split them up into two classes,
    - Initial State (IS) Radiative Collinear divergences.
    - Final State (FS) Radiative Collinear divergences.

Final State Collinear Divergence

- Just like for IR divergences the divergences arising from final state radiation will cancel with divergence's in the virtual term,

- We pick up a $1/\varepsilon$ pole from the phase space integration of the real piece and an identical piece (up to a sign) in the virtual amplitude,



KLN Theorem

- The KLN theorem tells us that all final state collinear divergences cancel when we sum over *degenerate states*.
- If we do not sum over all degenerate states then we will have left over divergence's.
- The answer we get will then not make sense!
- We can therefore only compute IR safe observables. i.e. observables where all IR singularities cancel.

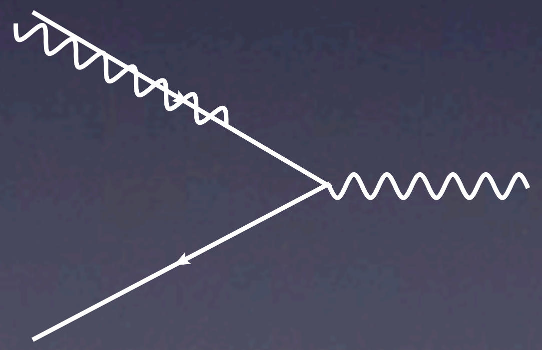
Infrared Finite Observables

- This means that we must be careful what we try to measure when we compare theory against experiment.
- Safe observables are generally,
  - Total cross sections.
  - Event Shapes.
  - Jets (with a good jet definition)

Initial state

- IR singularities from Initial state radiation are slightly different.

- They do not cancel between the real and virtual pieces.
- We do not sum over initial states of the form



- Instead these divergence's can be absorbed into the pdf's

Singularity Summary

- There are three kinds of singularity we encounter when performing NLO calculations.
  - UV singularities - Remove via renormalisation.
  - Final State IR singularities - Sum over degenerate states and combine the real and virtual contributions.
  - Initial State IR singularities - Absorb into the PDF's.

Summary

- We will finish our investigation into Renormalisation.
- Look at computing Next-to-Leading Order (NLO) Corrections.
- Understand IR singularities.

Lecture 10

- We will go through some of the modern techniques that are used to perform actual QCD computations.
- The spinor helicity formalism and helicity amplitudes.
- On-Shell Recursion Relations.
- Loops via Unitarity.

Helicity Amplitudes

- Usually prefer to compute helicity amplitudes,

$$A(p_1^{h_1}, p_2^{h_2}, p_3^{h_3}, \ldots, p_n^{h_n})$$

- Each external leg is describe in terms of its momenta and its helicity.
- We will assume we are dealing with massless particles, (but all the techniques are straightforwardly adaptable). So this is simply the spin of the associated external state.
- These can be separately squared and then integrated over the phase space.

Spinor-Helicity Method

- We will write the two component massless spinors as,

$$\langle 1^-| = \overline{u}_-(p_1), \ \ \langle 2^+| = \overline{u}_+(p_2)$$
$$|1^-\rangle = u_+(p_1), \ \ |2^+\rangle = u_-(p_2)$$

- Then the spinor products can be written as,

$$\overline{u}_-(k_1)u_+(k_2) = \langle 12\rangle \quad \overline{u}_+(k_1)u_-(k_2) = [12]$$

- These are anti-symmetric,

$$\langle 12\rangle = -\langle 21\rangle, \ \ [12] = -[21]$$

- We can connect these spinor products to the the Lorentz products,

$$\langle 12\rangle[21] = (p_1 + p_2)^2$$

Spinor-Helicity Method

- These spinor products can be viewed as “square roots” of the Lorentz products with a phase,

$$\langle 12\rangle = e^{-i\phi}\sqrt{(p_1+p_2)^2} \qquad [12] = e^{i\phi}\sqrt{(p_1+p_2)^2}$$

- The outer product can be written as,

$$|1\rangle\langle 1| + |1][1| = \not{p}_1$$

- We can use this identity to rewrite all momentum 4-vectors as spinors.
- We will see that we can express amplitudes in a more compact form if we do this.

Polarisation Vectors

- We need to write the polarisation vectors in terms of spinors as well.
- This can be done using,

$$\epsilon^{\mu}_{\pm}(p,n) = \pm \frac{\langle p^{\pm}|\gamma^{\mu}|n^{\pm}\rangle}{\langle p^{\mp}|n^{\pm}\rangle}$$

- The gauge choice for the polarisation vectors is taken into account by the arbitrary n vector.
- We can see that this representation satisfies the completeness relation

$$\sum_{\lambda=\pm} \epsilon^{*\mu}(p,\lambda)\epsilon^{\nu}(p,\lambda) = -g^{\mu\nu} + \frac{p^{\mu}n^{\nu} + p^{\nu}n^{\mu}}{p\cdot n}$$

Removing 4-Vectors

- Re-express common objects that we find in Feynman diagrams,

$$\overline{u}_-(p_1)\gamma^\mu u_-(p_2) = \langle 1|\gamma^\mu|2]$$

$$\overline{u}_-(p_1)\gamma^\mu\gamma^\nu u_-(p_2) = \langle 1|\gamma^\mu\gamma^\nu|2\rangle$$

- The gamma matrices will be contracted with some 4-vector, so we can remove all 4-vector terms,

$$p_{3\mu}\overline{u}_-(p_1)\gamma^\mu u_-(p_2) = \langle 1|\not{p}_3|2] = \langle 13\rangle[32]$$

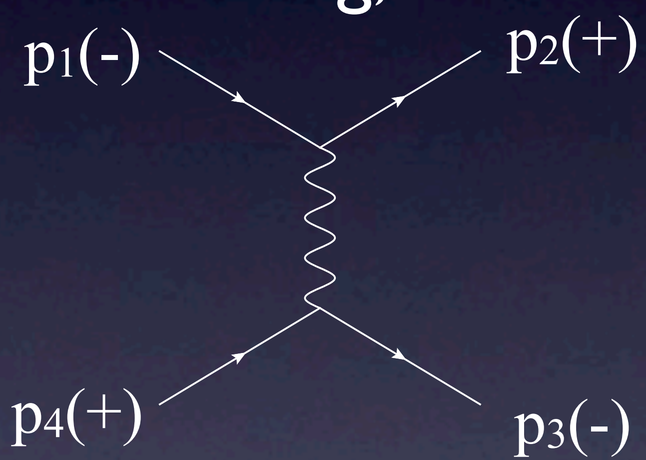
$$\overline{u}_-(p_1)\gamma^\mu u_-(p_2)\overline{u}_+(p_3)\gamma^\mu u_+(p_4) = \langle 14\rangle[32]$$

- There is also the very useful Schouten identity for manipulating these objects,

$$\langle ij\rangle\langle kl\rangle = \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle$$

An Example

- Let us rewrite one of our previous examples.
- We want to compute the helicity amplitudes, so first compute the amplitude, $A(1^-,2^+,3^-,4^+)$, we specific specific helicities for each leg,



- Only one Feynman diagram contributes as the other would be zero, (we saw that each diagram was separately gauge invariant earlier)

$$= -i(-ie)^2 \frac{[\overline{u}_-(p_1)\gamma^\mu u_+(p_2)][\overline{u}_+(p_4)\gamma_\mu u_-(p_3)]}{(p_2-p_1)^2}$$

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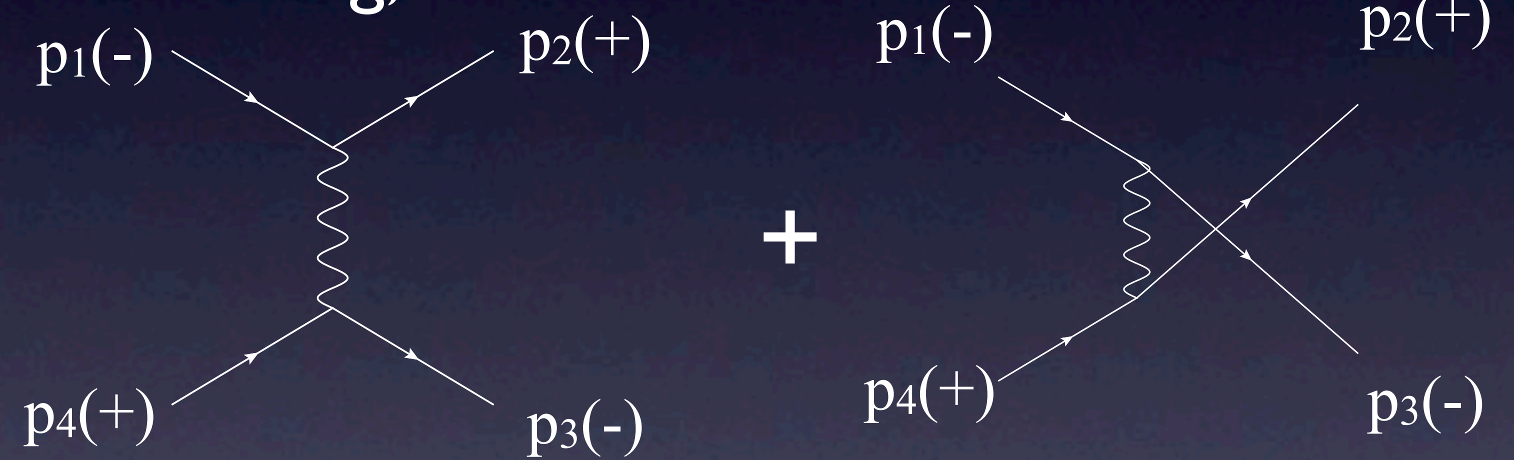

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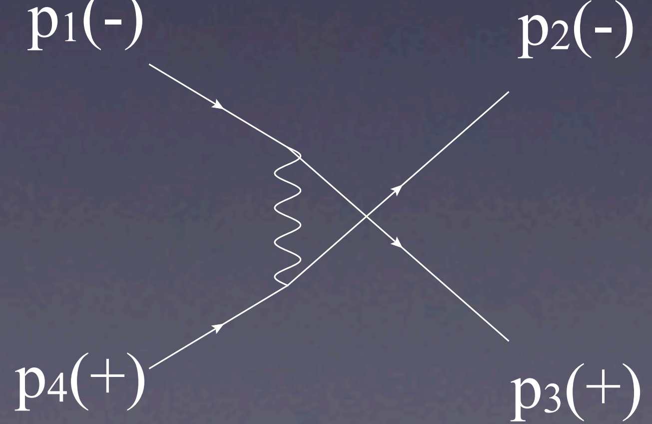
$$= -i(-ie)^2 \frac{[\overline{u}_-(p_1)\gamma^\mu u_+(p_2)][\overline{u}_+(p_4)\gamma_\mu u_-(p_3)]}{(p_2-p_1)^2}$$

- We can then rewrite the amplitude in spinor-helicity notation,

$$-i(-ie)^2 \frac{\langle 13\rangle[42]}{(p_2-p_1)^2} = -ie^2 \frac{\langle 13\rangle[42]}{\langle 12\rangle[21]}$$

- There is one other helicity amplitude to consider, $A(1^-,2^-,3^+,4^+)$, the rest are zero.



$$= i(-ie)^2 \frac{\langle 12\rangle[34]}{\langle 13\rangle[31]}$$

Amplitude Squared

- Apart from the compact expressions for each of the amplitudes there is another advantage.
- When we "square" the amplitude we have much less work to do,

$$|A|^2 = |A(1^-, 2^-, 3^+, 4^+)|^2 + |A(1^-, 2^+, 3^-, 4^+)|^2$$

- We can just directly square each helicity amplitude to get,

$$e^4 \left(\frac{\langle 12 \rangle [34][21] \langle 43 \rangle}{\langle 13 \rangle^2 [31]^2} + \frac{\langle 13 \rangle [24][31] \langle 42 \rangle}{\langle 12 \rangle^2 [21]^2} \right)$$

- This requires less work than dealing with the cross terms and traces of gamma matrices.

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$$e^4 \left(\frac{s_{12}s_{34}}{s_{13}^2} + \frac{s_{13}s_{24}}{s_{21}^2} \right)$$

- This requires less work than dealing with the cross terms and traces of gamma matrices.

Complexity of QCD Amplitudes

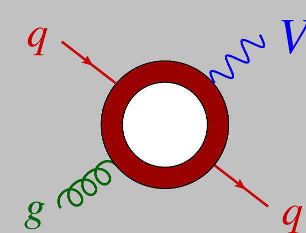
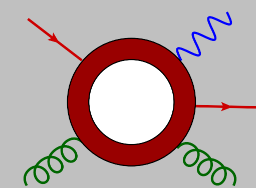
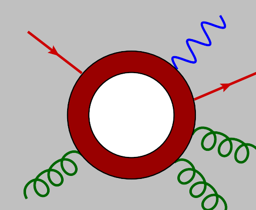
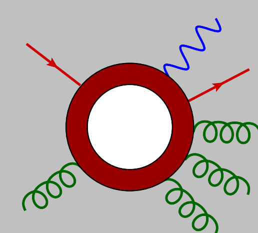
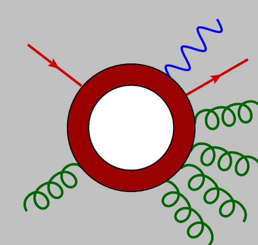
- In QCD we have quark-gluon, three gluon and four-gluon vertices.
- We need to consider all permutations over identical particles. This is particularly bad for high multiplicity gluon amplitudes.
- There is a factorial growth in the number of Feynman diagrams as we increase the number of legs.
- If we want to go beyond tree level this gets even worse.
- This makes the final amplitudes look very complicated.

All Gluon Amplitudes

- Lets count the number of diagrams we must include for a one-loop all gluon amplitude as we increase the number of legs.

#Legs	#Diagrams
6	~10,000
7	~150,000
8	~3,000,000
n	∞

V+Jets Amplitudes

1		11
2		110
3		1,253
4		16,648
5		256,265

Simple Amplitudes

- We might think that we are stuck with the difficult task of computing and combining large numbers of Feynman diagrams.
- But the final amplitudes are actually much simpler than we would expect.
- An example of this are the Parke-Taylor Amplitudes.
  - $A(1^+, 2^+, 3^+, \ldots, n^+) = 0$.
  - $A(1^-, 2^+, 3^+, \ldots, n^+) = 0$.
  - $A(1^-, 2^-, 3^+, \ldots, n^+)$.

MHV amplitudes

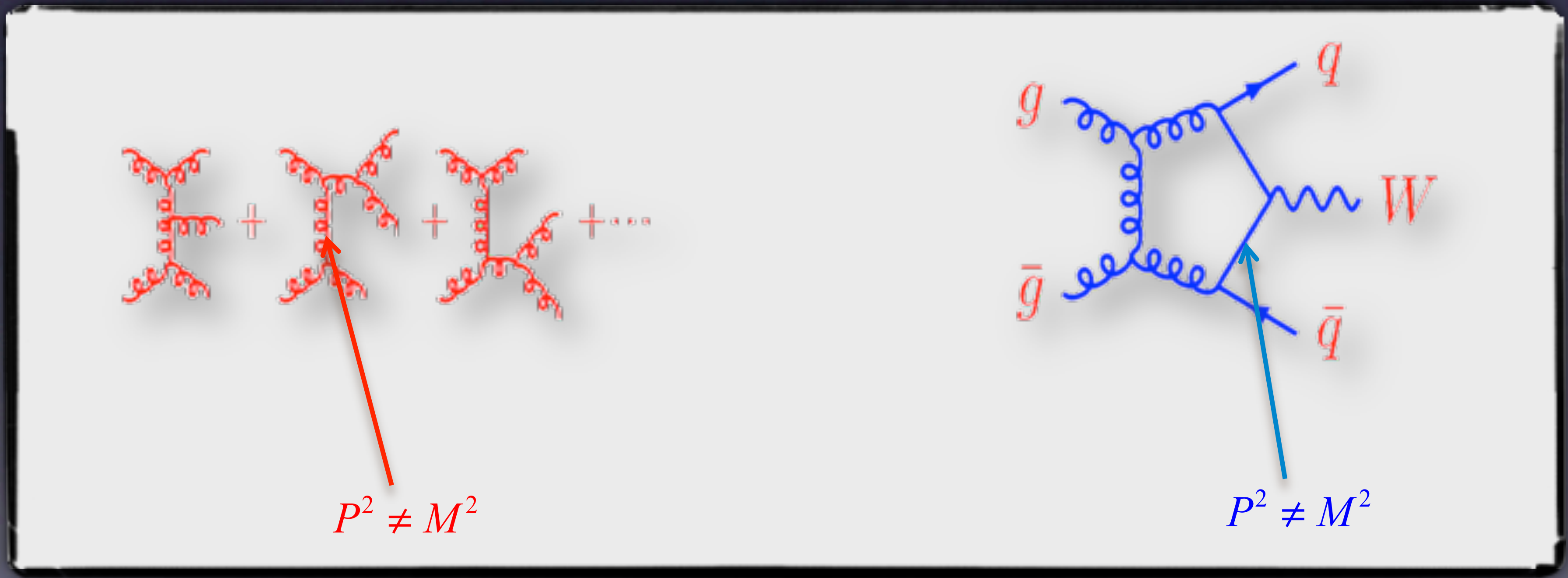
- These are three all-multiplicity amplitudes.
- If we were to compute them with Feynman diagrams we would need to sum together an infinite number of terms.
- The first two amplitudes are zero.
- The third is non-zero and is known as the Maximally-Helicity-Violating Amplitude,

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\ldots\langle n1\rangle}$$

Gauge Dependence

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\ldots\langle n1\rangle}$$

- Why is this amplitude so simple?
- Feynman Diagrams are a powerful tool but they do not take into advantage of all the symmetries of the system.
- The problem with Feynman Diagrams is that they are gauge dependant objects and they are built up from off-shell objects.

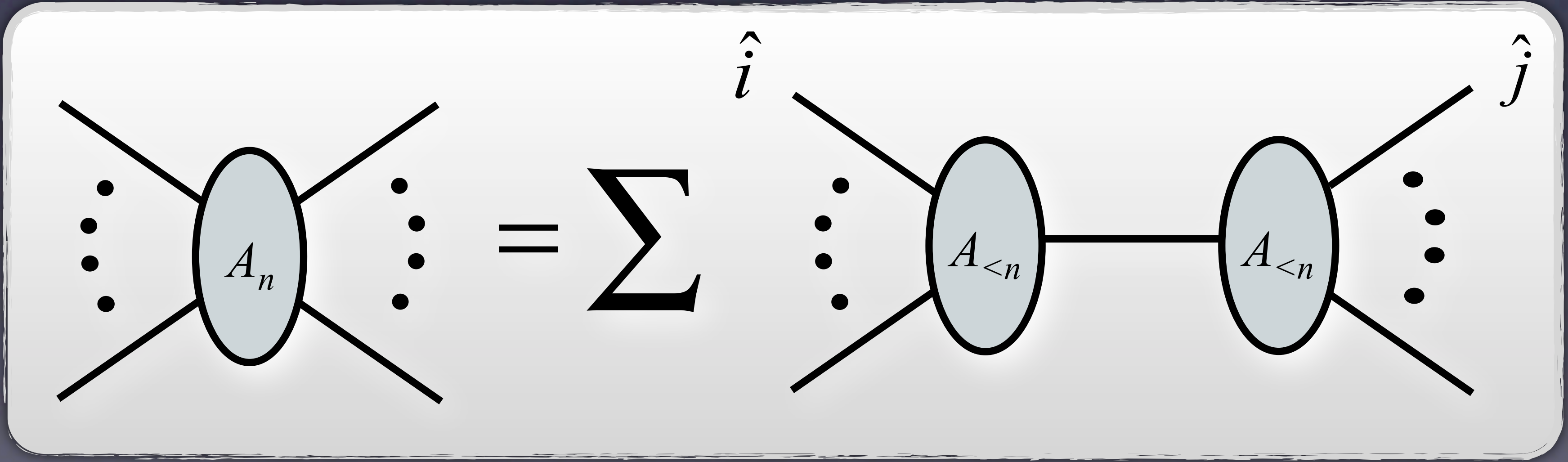


A Better Way

- The gauge dependance only cancels at the amplitude level.
- The final amplitudes are on-shell objects.
- A simple result after a very complicated computation procedure tells us that there is probably a better way.
- There is a better way, we should work with the amplitudes directly. They are,
    - On-shell.
    - Gauge invariant.
- They will therefore be much simpler.

On-shell Recursion

- How do we use amplitudes directly?
- On-shell recursion (BCFW) relations were discovered by Britto, Cachazo, Feng and Witten in 2005.
- Simple idea: build up amplitudes from amplitudes with fewer legs,



The Details

- How does this work?
- We pick two legs, i and j.
- We shift the momentum of these two legs so that
  - We conserve overall momentum in the amplitude.
  - The shifted legs remain on-shell.
- To do this we will need complex momentum (it is impossible otherwise).

The Shifted Momentum

- How can we shift these legs and satisfy these properties?
- We shift one of the spinor components of the momentum, this makes them complex momentum,

$$k_j^\mu = \langle j|\gamma^\mu|j] = \langle j|\gamma^\mu|j] - z\langle i|\gamma^\mu|j]$$
$$k_i^\mu = \langle i|\gamma^\mu|i] = \langle i|\gamma^\mu|i] + z\langle i|\gamma^\mu|j]$$

- We see that momentum is conserved and the momenta remain on-shell.

Recursion

- We then consider all divisions of the amplitude into two smaller amplitudes where one half contains leg i and the other leg j.
- Connecting each half of all such terms with a scalar propagator gives us the final amplitude.
- This connecting leg needs to be on-shell and so we fix z so that this is true.

Simple Example

- To make this clearer let us try a simple example.
- Let us compute the 6 point all gluon MHV amplitude $A(1^-,2^-,3^+,4^+,5^+,6^+)$.
- This is a relatively complicated amplitude to compute using Feynman diagrams.
- How do we start?

Simple Example

- Pick two legs to "shift".
- We will pick leg 2 and leg 3 so their momenta become,

$$k_2^\mu = \langle 2|\gamma^\mu|2] = \langle 2|\gamma^\mu|2] + z\langle 2|\gamma^\mu|3]$$
$$k_3^\mu = \langle 3|\gamma^\mu|3] = \langle 3|\gamma^\mu|3] - z\langle 2|\gamma^\mu|3]$$

- The value of z will depend on how we split the amplitude up.
- Next we look at the ways we can split the amplitude up.

Simple Example

- We get six possible terms, five of which vanish.

$$A(1^-,\hat{2}^-,-\hat{P}_{12}^{\pm})\frac{1}{(p_1+p_2)^2}A(\hat{P}_{12}^{\mp},\hat{3}^+,4^+,5^+,6^+)$$

$$A(6^+,1^-,\hat{2}^-,-\hat{P}_{126}^{\pm})\frac{1}{(p_1+p_2+p_6)^2}A(\hat{P}_{126}^{\mp},\hat{3}^+,4^+,5^+)$$

$$A(5^+,6^+,1^-,\hat{2}^-,\hat{P}_{34}^{\pm})\frac{1}{(p_3+p_4)^2}A(-\hat{P}_{34}^{\mp},\hat{3}^+,4^+)$$

- As a number of the amplitudes vanish we are left with,

$$A(5^+,6^+,1^-,\hat{2}^-,\hat{P}_{34}^{+})\frac{1}{(p_3+p_4)^2}A(-\hat{P}_{34}^{-},\hat{3}^+,4^+)$$

The Amplitude

- Each of the remaining amplitudes is an MHV amplitude so we can write down expressions for them,

$$A(5^+, 6^+, 1^-, \hat{2}^-, \hat{P}_{34}^+) = \frac{\langle 1\hat{2}\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 1\hat{2}\rangle\langle \hat{2}\hat{P}_{34}\rangle\langle \hat{P}_{34}5\rangle}$$

$$A(-\hat{P}_{34}^-, \hat{3}^+, 4^+) = \frac{[\hat{3}4]^4}{[\hat{P}_{34}\hat{3}][\hat{3}4][4\hat{P}_{34}]}$$

- We can now set z as it is chosen so that $\hat{P}_{34}$ remains on-shell. This constraint gives us,

$$z = \frac{(p_3 + p_4)^2}{\langle 2|\not{p}_3 + \not{p}_4|3]}$$

Simplifying

- As we have shifted only one of the spinor components in each momentum then we can simplify these expressions

$$A(5^+, 6^+, 1^-, \hat{2}^-, \hat{P}_{34}^+) = \frac{\langle 12\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 12\rangle\langle 2\hat{P}_{34}\rangle\langle \hat{P}_{34}5\rangle}$$

$$A(-\hat{P}_{34}^-, \hat{3}^+, 4^+) = \frac{[34]^4}{[\hat{P}_{34}3][34][4\hat{P}_{34}]}$$

- The only remaining shifted momentum is given by,

$$\hat{P}_{34} = p_3 + p_4 + \frac{(p_3 + p_4)^2}{\langle 2|\not{p}_3 + \not{p}_4|3]}\langle 2|\gamma^\mu|3]$$

The Final Step

- Multiply the two amplitudes together and the propagator,

$$\frac{\langle 12\rangle^4}{\langle 56\rangle\langle 61\rangle\langle 12\rangle\langle 2\hat{P}_{34}\rangle\langle \hat{P}_{34}5\rangle}\frac{1}{\langle 43\rangle[34]}\frac{[34]^4}{[\hat{P}_{34}3][34][4\hat{P}_{34}]}$$

- We can use two simple identities to simplify this,

$$[4\hat{P}_{34}]\langle \hat{P}_{34}5\rangle = [4|\not{p}_3 + p_4|5\rangle = [43]\langle 45\rangle$$
$$\langle 2\hat{P}_{34}\rangle[\hat{P}_{34}3] = \langle 2|\not{p}_3 + p_4|3] = \langle 24\rangle[43]$$

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The Final Step

- Multiply the two amplitudes together and the propagator,

$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle}$$

- We can use two simple identities to simplify this,

$$[4\hat{P}_{34}]\langle\hat{P}_{34}5\rangle = [4|\not{p}_3 + p_4|5\rangle = [43]\langle 45\rangle$$
$$\langle 2\hat{P}_{34}\rangle[\hat{P}_{34}3] = \langle 2|\not{p}_3 + p_4|3] = \langle 24\rangle[43]$$

On-Shell 3-Point Vertex

- In this example we required an on-shell three point amplitude.
- How can such an object exist?
- Momentum conservation would tell us that,

$$(p_1 \cdot p_2) = (p_2 \cdot p_3) = (p_3 \cdot p_1) = 0$$

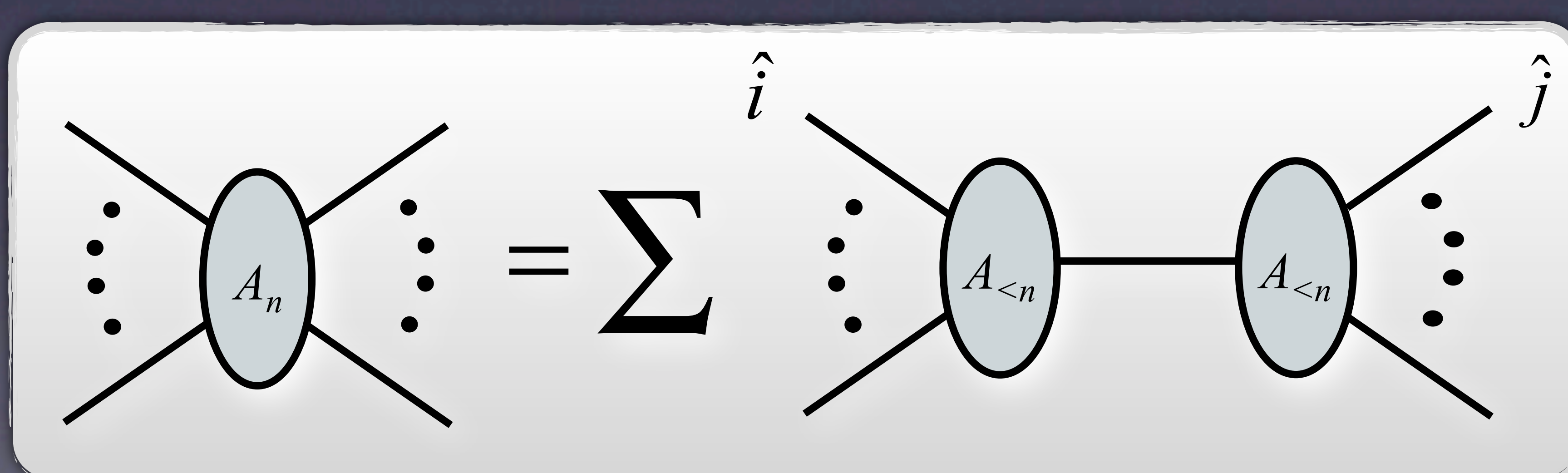
- We are using complex momentum so this is no longer the case!

$$\langle 12 \rangle \not\propto [12]$$

- For real momenta these are proportional and so there are no non-zero invariants we could use to build a vertex.
- We can build up all amplitudes from just the complex three-point ones, even though the QCD Lagrangian contains a 4-point interaction term.

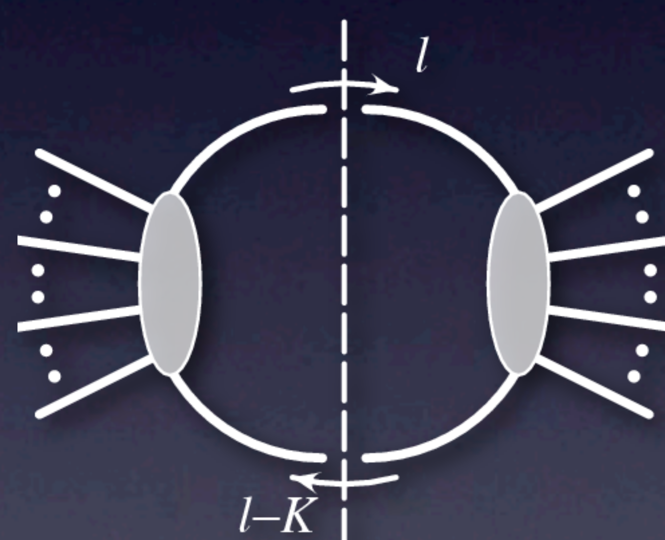
On-Shell Summary

- At tree level we can use on-shell recursion to very easily build up amplitudes that would be difficult using Feynman diagrams.
- To prove these relations we need only use complex momenta, some complex analysis and the simple properties of the amplitudes.
- This provides us with a very powerful technique.



Loops

- At the loop level we will use unitarity.
- We will glue tree amplitudes together to get loops,



- As we now have a method for producing compact trees we will be also be able to produce compact loops.

Summary

- We have introduced the spinor-helicity technique as an efficient way of computing amplitudes.
- We have seen how we can reduce the complicated sum of Feynman diagrams down to much simpler amplitudes.
- We have seen how simple factorisation and complex analysis give us a very powerful techniques for computing amplitudes.