# Introduction to Quantum Field Theory and QCD Lecture 7 \& 8 

## Darren Forde CERN \& NIKHEF

BND Summer School 2010 Oostende, Belgium, Sept 6-17th 2010

## Lecture 7

- Investigate how non-abelian local symmetries introduce new interaction terms.
- Examine how a local $\operatorname{SU}(\mathrm{N})$ gauge symmetry leads to QCD.
- Investigate some of the properties of QCD computations including the use of PDF's.


## Yang-Mills Theory

- QED is an abelian $U(I)$ theory, its Lagrangian is invariant under local phase rotations.
- Consider a more general class of theories.
- Want to investigate invariance under any continuous symmetry group.
- Theories of this type are known as Yang-Mills theories.
- QCD is an example of this where we demand invariance under rotations in an internal SU(3) space.


## The General Case

- In QED we had a single field transforming.
- Consider generalising to $N$ copies of the field,

$$
\Psi_{a}(x)=\left(\begin{array}{c}
\Psi_{1}(x) \\
\Psi_{2}(x) \\
\cdots \\
\Psi_{N}(x)
\end{array}\right)
$$

- The field now carries an additional index, $a$.
- This internal space has dimension $N$ and $a$ runs from 1 to $N$.
- We want to consider local invariant transforms of the field with this additional index.


## General Transformation

- A general local transformation of the field in this internal space is written as,

$$
\Psi_{a}(x) \rightarrow U_{a b}(x) \Psi_{b}(x)
$$

- We will consider local phase transformations, so $U(x)$ is of the general form,

$$
U_{a b}(x)=\exp \left[i \sum_{i} \alpha^{i}(x) t_{a b}^{i}\right]
$$

- The vector of fields $\Psi_{a}$ will multiply (products) of the $t_{i}$ which are $N \times N$ matrices,

$$
\left(\begin{array}{cccc}
\cdots & & & \\
& \cdots & & \\
& & \cdots & \\
& & & \cdots
\end{array}\right)\left(\begin{array}{c}
\Psi_{1}(x) \\
\Psi_{2}(x) \\
\cdots \\
\Psi_{N}(x)
\end{array}\right)
$$

## The Symmetry Group <br> $$
U_{a b}(x)=\exp \left[i \sum_{i} \alpha^{i}(x) t_{a b}^{i}\right]
$$

- What form can the $t_{i}$ matrices take?
- We will choose the $t_{i}$ to be the generators of a symmetry group, there will be as many $\alpha_{i}{ }^{\text {'s }}$ as there are generators.
- To connect with QCD we will choose this group to be the $\operatorname{SU}(N)$ Lie Algebra.
- In QED we had a single generator, a constant.
- Here the number of generators will depend upon the representation of the group.
- In general the $t_{i}$ generators will not commute.


## SU(N) Lie Algebras

- We now need to go into a little more detail about Lie Algebras.
- $\operatorname{SU}(N)$ is the group of all unitary $N \times N$ matrices with determinant equal tol.
- The generators will satisfy commutation relations related to their group structure.
- The commutation relations of the Lie Algebra we use will define the structure of the Lagrangian,

$$
\left[t^{A}, t^{B}\right]=i f^{A B C} t^{C}
$$

- Theories which are defined through such Lie Algebras are known as non-abelian.


## SU(N)

- To proceed we need to have a representation of the group.
- We will start by using the Fundamental representation, e.g. for $\mathrm{SU}(3)$ this is

$$
\begin{gathered}
t^{1}=\frac{1}{2}\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) t^{2}=\frac{1}{2}\left(\begin{array}{ccc}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{array}\right) t^{3}=\frac{1}{2}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right) \\
t^{4}=\frac{1}{2}\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right) t^{5}=\frac{1}{2}\left(\begin{array}{ccc}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{array}\right) t^{6}=\frac{1}{2}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right) \\
t^{7}=\frac{1}{2}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right) t^{8}=\frac{1}{2 \sqrt{3}}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right) \\
8
\end{gathered}
$$

## Fundamental Representation

- There are $N^{2}-1$ generators which for $N=3$ gives the $8 t^{i}$ s we have written down.
- As we would expect these matrices satisfy the expected commutation relations,

$$
\left[t^{A}, t^{B}\right]=i f^{A B C} t^{C}
$$

- The $f^{A B C}$ are the structure constants of the group.
- We have normalised these generators so that,

$$
\operatorname{Tr} t^{A} t^{B}=T_{R} \delta^{A B}, T_{R}=\frac{1}{2}
$$

## Adjoint Representation

- An alternative representation that will be useful is the Adjoint representation.
- This is built using the structure constants in the following way,

$$
\left(T^{A}\right)_{B C}=-i f^{A B C}
$$

- Again this representation of the generators satisfies a similar commutation relation,

$$
\left[T^{A}, T^{B}\right]=i f^{A B C} T^{C}
$$

- As well as,

$$
\operatorname{Tr} T^{C} T^{D}=\sum_{A, B} f^{A B C} f^{A B D}=C_{A} \delta^{C D}, C_{A}=N
$$

- Also we have the Casimir Operator,

$$
\sum_{A} t_{a b}^{A} t_{b c}^{A}=C_{F} \delta_{a c}, C_{F}=\frac{N^{2}-1}{2 N}
$$

## Local Invariance

- In the abelian case we were forced to introduce a new field (the photon) that transformed in a certain way under the local gauge transformation and introduce a covariant derivative to guarantee local invariance.
- In the non-abelian case we will have to introduce one new field for each generator of the symmetry group.
- For QCD $N=3$ and so there are 8 generators and hence 8 new fields.
- These will be the gluon fields.


## The Gluon Fields

- These new gluon fields are introduced into the covariant derivative as,

$$
D_{\alpha}=\left(\partial_{\alpha}+i g \sum_{A} t^{A} A_{\alpha}^{\prime A}\right)
$$

- Each gluon field $A^{A}$ is multiplied by a generator of the Lie Group in the Fundamental representation.
- We also have what will become the coupling constant $g$.
- After we introduce the kinetic term for the fields we will see that these really are gluons.


## Covariant Derivative

$$
D_{\alpha}=\left(\partial_{\alpha}+i g \sum_{A} t^{A} A_{\alpha}^{\prime} A\right)
$$

- The quarks are spinors and so we want the covariant derivative of them to transform under the local transformation in the same way,

$$
D_{a} \Psi(x) \rightarrow D_{a}^{\prime} \Psi^{\prime}(x)=U(x) D_{a} \Psi(x)
$$

- Performing this transformation we have,

$$
\begin{aligned}
D_{\alpha}^{\prime} \Psi^{\prime}(x) & =\left(\partial_{\alpha}+i g \sum_{A} t^{A} A_{\alpha}^{\prime} A\right) U(x) \Psi(x) \\
& =\left(\partial_{\alpha} U(x)\right) \Psi(x)+U(x) \partial_{\alpha} \Psi(x)+i g \sum_{A} t^{A} A_{\alpha}^{\prime} A^{A} U(x) \Psi(x)
\end{aligned}
$$

## Covariant Derivative

$$
D_{\alpha}=\left(\partial_{\alpha}+i g \sum_{A} t^{A} A_{\alpha}^{\prime} A\right)
$$

- The quarks are spinors and so we want the covariant derivative of them to transform under the local transformation in the same way,

$$
D_{a} \Psi(x) \rightarrow D_{a}^{\prime} \Psi^{\prime}(x)=U(x) D_{a} \Psi(x)
$$

- Performing this transformation we have,

$$
\begin{aligned}
D_{\alpha}^{\prime} \Psi^{\prime}(x) & =\left(\partial_{\alpha}+i g \sum_{A} t^{A} A_{\alpha}^{\prime A}\right) U(x) \Psi(x) \\
& =\left(\partial_{\alpha} U(x)\right) \Psi(x)+U(x) \partial_{\alpha} \Psi(x)+i g \sum_{A} t^{A} A_{\alpha}^{\prime A} U(x) \Psi(x)
\end{aligned}
$$

## Gluon Transformation

- To satisfy this transformation property then we must demand that the gluon fields, $A^{4}$, transform to cancel these additional terms,

$$
\sum_{A} t^{A} A_{\alpha}^{\prime} A=\sum_{A} t^{A} U(x) A_{\alpha}^{A} U^{-1}(x)+\frac{i}{g}\left(\partial_{\alpha} U(x)\right) U^{-1}(x)
$$

- Using this transformation we see that the extra terms will cancel,

$$
\begin{aligned}
& i g \sum_{A} t^{A} A_{\alpha}^{\prime A} U(x) \Psi(x)=i g \sum_{A} t^{A} U(x) A_{\alpha}^{A} U^{-1}(x) U(x) \Psi(x) \\
& +(i g) \frac{i}{g}\left(\partial_{\alpha} U(x)\right) U^{-1}(x) U(x) \Psi(x)
\end{aligned}
$$

$$
\begin{aligned}
\left(\partial_{\alpha} U(x)\right) \Psi(x)+U(x) \partial_{\alpha} \Psi(x)+ & i g \sum_{A} t^{A} A_{\alpha}^{\prime} A U(x) \Psi(x) \\
& =U(x)\left(\partial_{\alpha}+i g \sum_{A} t^{A} A_{\alpha}^{\prime A}\right) \Psi(x)
\end{aligned}
$$

## Kinetic Terms

- Again we have introduced interacting fields but not defined a kinetic term for them.
- As in QED we can only pick a kinetic term that transforms in the correct way under the gauge transformation.
- The only such object is written as,

$$
-\frac{1}{4} F^{A \mu \nu} F_{\mu \nu}^{A}
$$

- The non-abelian field strength tensor is given by,

$$
F_{\mu \nu}^{A}(x)=\partial_{\mu} A_{\nu}^{A}(x)-\partial_{\nu} A_{\mu}^{A}(x)+g f^{A B C} A_{\mu}^{B}(x) A_{\nu}^{C}(x)
$$

## $F^{\mu \nu}$ Transformations

- The field strength tensor will also transform under the gauge transformation,

$$
F_{\mu \nu}^{\prime A}(x)=U(x) F_{\mu \nu}^{A}(x) U^{-1}(x)
$$

- These results can also be derived from the identity,

$$
\left[D_{\mu}, D_{\nu}\right]=i g t^{A} F_{\mu \nu}^{A}
$$

- We cannot write down an invariant term for the mass and so the gluon fields are massless.
- For example the following is not invariant,

$$
m^{2} A^{\mu} A_{\mu}
$$

## QCD

- Combining all these pieces together we get the QCD Lagrangian written in a compact form,

$$
\mathcal{L}_{\mathrm{QCD}}=-\frac{1}{4}\left(F_{\mu \nu}^{b}\right)^{2}+\sum_{f} \bar{\Psi}_{a, f}(i \not D-m) \Psi_{a, f}
$$

- We see that the quarks carry an index, $a$, with a running over $1,2,3$.
- This labels the colour each quark now carries.
- The quarks also have a flavour index as there is one quark field for each flavour $f=u, d, c, s, b, t$.


## Quark-Gluon Interactions

- What interactions does this Lagrangian contain?
- Let us first look at the coupling of two quarks to a gluon.
- This term comes from the covariant derivative,

$$
i \bar{\Psi} \not D \Psi=i \bar{\Psi} \not \partial \Psi-g \sum_{A} t^{A} \bar{\Psi} A^{A} \Psi
$$

Colour Flow

- We have dropped the colour index from the gluon and the quark.


## Gluon-Gluon Interactions

- One on the defining features of a nonabelian gauge theory is the presence of interaction terms between the gauge bosons.
- Unlike in QED we will have interactions between the gluons.
- These interactions come from the additional term in the Field strength tensor.

$$
F_{\mu \nu}^{A}(x)=\partial_{\mu} A_{\nu}^{A}(x)-\partial_{\nu} A_{\mu}^{A}(x)+g f^{A B C} A_{\mu}^{B}(x) A_{\nu}^{C}(x)
$$

## Three Gluon Interactions

- In the $F^{\mu v} F_{\mu v}$ term we will have pieces of the type,

$$
g f^{A B C}\left(\partial_{\mu} A_{\nu}^{A}(x)\right) A^{\mu B} A^{\nu C}
$$

- So we have three gluon fields interacting at a point with a coupling constant g.
- This coupling is also multiplied by an fabc structure constant, and so the colour flow will be,



## Four Gluon Interactions

- There is a second type of term that will arise from the $F^{\mu v} F_{\mu v}$ term,

$$
g^{2} f^{A B C} f^{C D E} A^{\mu B} A^{\nu C} A_{\mu}^{D} A_{\nu}^{E}
$$

- Here we have four gluon fields interacting at a point with a coupling constant $g^{2}$.
- This coupling is also multiplied by an fabcfcde structure constant, and so the colour flow will be,



## Final Points

$$
\mathcal{L}_{\mathrm{QCD}}=-\frac{1}{4}\left(F_{\mu \nu}^{b}\right)+\sum_{f} \bar{\Psi}_{a, f}(i \not D-m) \Psi_{a, f}+\mathcal{L}_{\mathrm{gauge}}+\mathcal{L}_{\mathrm{ghost}}
$$

- We can now proceed to quantise this theory.
- To do this we will have to add a gauge fixing term as we did in QED.
- So our Feynman diagrams are going to be gauge dependant.
- In general we must also add in a term for what are called "ghost" fields these are required to preserve Unitarity when we quantise the theory.
- After quantisation we can write down a number of Feynman rules.


## Feynman Rules

- The Feynman rules for propagators in QCD are given by,

$$
\begin{aligned}
a \underset{\mu}{\text { mom }}{ }_{\nu}^{b} & =\frac{-i g^{\mu \nu}}{k^{2}+i \epsilon} \delta_{a b} \\
i \xrightarrow{k} j & =\frac{i(k+m)}{k^{2}-m^{2}+i \epsilon} \delta_{i j}
\end{aligned}
$$

## Feynman Rules

- The Feynman rules for the vertices are given by,


$$
\begin{aligned}
=-g f^{A B C} & \left(g^{\mu \nu}(p-q)^{\rho}\right. \\
+ & g^{\nu \rho}(q-r)^{\mu} \\
& \left.+g^{\rho \mu}(r-p)^{\nu}\right)
\end{aligned}
$$

## Feynman Rules

- The Feynman rules for the vertices are given by,


$$
\begin{aligned}
= & -i g^{2} f^{a b e} f^{e c d}\left(g^{\rho \nu} g^{\mu \sigma}-g^{\rho \sigma} g^{\mu \nu}\right) \\
& -i g^{2} f^{a c e} f^{e b d}\left(g^{\rho \mu} g^{\nu \sigma}-g^{\rho \sigma} g^{\mu \nu}\right) \\
& -i g^{2} f^{a d e} f^{e b e}\left(g^{\rho \nu} g^{\mu \sigma}-g^{\rho \mu} g^{\sigma \nu}\right)
\end{aligned}
$$

## External States

- For fermions external states are given by,

$$
\begin{aligned}
& \xrightarrow{p \rightarrow}=u(p) \text { for incoming quarks } \\
& \xrightarrow{p_{\leftarrow}}=\bar{u}(p) \text { for outgoing quarks } \\
& \xrightarrow[p_{\leftarrow}]{\stackrel{p}{\longrightarrow}} \\
& =v(p) \text { for incoming antiquarks } \\
& =\bar{v}(p) \text { for outgoing quarks }
\end{aligned}
$$

- For gluon external states,



## External States

- For fermions external states are given by,

$$
\begin{aligned}
& \xrightarrow{p \rightarrow}=u(p) \text { for incoming quarks } \\
& \xrightarrow{p \longleftarrow}=\bar{u}(p) \text { for outgoing quarks } \\
& \xrightarrow[p_{\leftarrow}]{\stackrel{p}{\longrightarrow}} \\
& =v(p) \text { for incoming antiquarks } \\
& =\bar{v}(p) \text { for outgoing quarks }
\end{aligned}
$$

- For gluon external states,



## QCD Computations

- Computations seem similar to QED, draw all Feynman diagrams and compute etc.
- QCD Feynman diagrams are similar to QED diagrams but we now have colour information flowing through them.
- Unlike QED with electrons and photons where they can be observed we never see quarks and gluons.
- Quarks and Gluons are confined.
- What we see a spectrum of bound state particles, such as Pion's etc.


## DIS

- The process $e^{-} p^{+} \rightarrow e^{-}+X$ is known as Deeply Inelastic Scattering (DIS)

- The electron interacts via a photon with the proton.
- We can derive the contribution from this part using the Feynman rules for QED.
- Which one of the partons (quarks) in the proton does the photon interact with?


## Partons

- Assume that the proton is made up of a noninteracting set of partons.
- Each parton contains a fraction $x_{i}\left(0<x_{i}<1\right)$ of the total proton momentum.
- The probability of "finding" parton $i$ in the proton with a momentum between $x_{i}$ and $x_{i}+d x_{i}$ is then given by,

$$
f_{i / P}\left(x_{i}\right) d x_{i}
$$

- The function $f_{i / P}\left(x_{i}\right)$ is known as a "parton distribution function" or "parton density"


## Parton Distribution Functions

- They contain all the complicated information about the behaviour of the partons in the protons.
- Commonly known as "Soft" physics as it occurs at much lower energies than the collision.
- There are PDF's for every flavour of quark as well as for gluons (which can also be present).
- Each PDF is universal and independent of the scattering process.


## Computing PDF's

- We do not know how to construct them from first principles as they depend upon non-perturbative physics.
- Use experimental data combined with an ansatz for their structure to compute them. [MRST, CTEQ]

$$
x f_{i / P}(x)=A_{a} x^{\Delta_{a}}(1-x)^{\eta_{a}}\left(1+\epsilon_{a} \sqrt{x}+\gamma_{a} x\right)
$$

- Some newer approaches use neural networks. [NNPDF]
- Different experiments provide data in different regions of $x$.


## Parton Distribution Functions

- For example the MSTW 2008 pdf set at NLO looks like,



## Summary

- We have seen how non-abelian local symmetries introduce new interaction terms.
- We have seen how a local $\operatorname{SU}(\mathrm{N})$ gauge symmetry leads to QCD.
- We have started investigating the properties of QCD computations including the use of PDF's.


## Lecture 8

- Learn about the Factorisation theorem and how this allows us to compute QCD amplitudes.
- Go through a simple example, including computing the cross section.
- Asymptotic Freedom.
- Investigate the properties of higher order quantum predictions.


## DIS

- The process $e p^{+} \rightarrow e^{-}+X$ is known as Deeply Inelastic Scattering (DIS)

- We can derive the contribution from this part using the Feynman rules for QED.
- The quark PDF is used to compute which quark interacts with the photon.


## Splitting the Computation

- We can write the cross section computation as
$\sigma\left(e^{-}(k) p(P) \rightarrow e^{-}\left(k^{\prime}\right)+X\right)=\int_{0}^{1} d x \sum_{f} f_{f / P}(x) \sigma\left(e^{-}(k) q_{f}(x P) \rightarrow e^{-}\left(k^{\prime}\right)+q_{f}\left(P^{\prime}\right)\right)$
- This splits off the soft QCD physics related to a quark in the proton.
- The remaining piece is computed as though the quark was a free particle with a modified momentum.
- $\quad \sigma\left(e^{-}(k) q_{f}(x P) \rightarrow e^{-}\left(k^{\prime}\right)+q_{f}\left(P^{\prime}\right)\right)$ is therefore the hard scattering cross section.
- Can we justify this division which is based (in part) on the assumption that the partons in the proton are noninteracting?


## Infinite Momentum Frame

- One of our assumptions when computing the scattering process is that at infinity the fields become well separated.
- How does this assumption apply when our final state particles are confined?
- To understand this we need to select a reference frame that simplifies the problem we are looking at.
- Rather than computing in the lab frame we switch to the proton "Infinite momentum frame".


## "Stationary" Partons

- From the point of view of the interacting photon interactions in the proton will be time-dilated by a factor of,

$$
\gamma=\frac{1}{\sqrt{1-v_{\text {proton }}^{2} / c^{2}}}
$$

- Furthermore if we were to imagine the proton having a radius in its rest frame of $r_{\text {proton }}$ then from the point of view of the photon the proton is Lorentz contracted into a "pancake" of thickness $2 r_{\text {proton }} / \gamma$.

- To the photon the proton looks like a non-interacting collection of stationary partons.


## Asymptotic Freedom

- Why do the particles in the proton behave as though they were free at high energies while they remain bound into hadrons at low energies?
- Asymptotic Freedom.
- In the early 70's it was discovered that coupling constants were not constants!

$$
g \rightarrow g(\hat{\mu}) \quad \text { This is the renormalisation scale } \text { will discuss in the next }
$$

- Typically in theories like QED the interaction grew stronger as the energy scale increased.
- For QCD though the interaction grows weaker as the energy scale is increased.


## Scale Dependance

- Schematically for QED we have, $g(\mu)$


High

## Scale Dependance

- Schematically for QED we have, $g(\mu)$



## A Factorised Result

$$
\sigma\left(e^{-}(k) p(P) \rightarrow e^{-}\left(k^{\prime}\right)+X\right)=\int_{0}^{1} d x \sum_{f} f_{f / P}(x) \sigma\left(e^{-}(k) q_{f}(x P) \rightarrow e^{-}\left(k^{\prime}\right)+q_{f}\left(P^{\prime}\right)\right)
$$

- Our computation now reduces to finding the hard scattering cross section.
- Using the parton model we assume that the states were initially well separated and we will assume that they will be again in the far future so that we can use Feynman diagram techniques.
- The hard scattering is then computable directly using Feynman diagrams.
- The in and out states are made up of familiar quarks, gluons, photons and electrons.


## Hadron Collisions

- Consider the collision of two protons.
- Let us assume that we can generalise our previous result for the differential cross section,

$$
d \sigma_{A B \rightarrow F+X}\left(p, p^{\prime}\right)=\sum_{\text {partons i,j }} \int_{0}^{1} d x d x^{\prime} f_{i / A}(x) f_{j / B}(x)
$$

- This can be visualised as



## Factorisation

- Can we justify this? Can we really separate two PDF's from the hard scattering?
- We now have two complicated protons that are colliding.
- A more sophisticated argument will be needed than for DIS, where we could go to the infinite momentum frame for the electron.

- After careful analysis and use of gauge theory we can show this is true this is the QCD Factorisation Theorem.


## Factorisation

- The differential cross section can be written as (additional terms are suppressed by inverse powers of the hard scale.)

$$
d \sigma_{A B \rightarrow F+X}\left(p, p^{\prime}\right)=\sum_{\text {partons } \mathrm{i}, \mathrm{j}} \int_{0}^{1} d x d x^{\prime} f_{i / A}\left(x, \mu_{F}\right) f_{j / B}\left(x, \mu_{F}\right)
$$

- We have introduced a factorisation scale $\mu F$.
- We can "view" this as the energy scale below which an interaction would be included in the PDF rather than the hard scattering element.
 u

- The PDF's and the hard scattering cross section both have a dependance on this scale, which we can compute.


## An Example: Drell-Yan

- As an example we will work out the cross section in detail for the Drell-Yan process at leading order,

- To simplify things we compute what would be a photon propagator as though it was an external particle.


## The Amplitude

- The Feynman diagram for this process gives,

$$
A=-i e\left(\bar{v}_{s_{2}}\left(p_{2}\right) \gamma^{\mu} u_{s_{1}}\left(p_{1}\right)\right) \epsilon_{\mu}^{*}(k, \lambda)
$$

- We want to compute $|A|^{2}$ so we also need,

$$
A^{*}=A^{\dagger}=i e\left(\bar{u}_{s_{1}}\left(p_{1}\right) \gamma^{\nu} v_{s_{2}}\left(p_{2}\right)\right) \epsilon_{\nu}(k, \lambda)
$$

- Next we will combine these two expressions.


## The Amplitude ${ }^{2}$

- We will contract all vectors and spinor chains when we "square" the amplitude,

- Summing over both photon and spinor helicities we have,
$\sum_{s_{1} s_{2} \lambda}|A|^{2}=\sum_{s_{1} s_{2} \lambda} e^{2}\left(\bar{v}_{2} \gamma^{\mu} u_{1}\right)\left(\bar{u}_{1} \gamma^{\nu} v_{2}\right) \epsilon_{\mu}^{*}(k, \lambda) \epsilon_{\nu}(k, \lambda)$


## The Details

- Now use our understanding of spinors and polarisation tensors to simplify this,


$$
=e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma^{\nu}\right]\left(-g_{\mu \nu}+\frac{k_{\mu} k_{\nu}}{k^{2}}\right)
$$

## The Result

- Contracting the Lorentz indices gives,

$$
\begin{aligned}
& =e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma^{\nu}\right]\left(-g_{\mu \nu}+\frac{k_{\mu} k_{\nu}}{k^{2}}\right) \\
& =-e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma_{\mu}\right]+\frac{e^{2}}{k^{2}} \operatorname{Tr}\left[\not p_{2} k \not k p_{1} \not k\right]
\end{aligned}
$$

## The Result

- Contracting the Lorentz indices gives,

$$
\begin{aligned}
& =e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma^{\nu}\right]\left(-g_{\mu \nu}+\frac{k_{\mu} k_{\nu}}{k^{2}}\right) \\
& =-e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma_{\mu}\right]+\frac{e^{2}}{k^{2}} \operatorname{Tr}\left[\not \psi_{2} k_{2} \not p_{1} k_{k}\right]
\end{aligned}
$$

## The Result

- Contracting the Lorentz indices gives,

$$
\begin{aligned}
& =e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma^{\nu}\right]\left(-g_{\mu \nu}+\frac{k_{\mu} k_{\nu}}{k^{2}}\right) \\
& =-e^{2} \operatorname{Tr}\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma_{\mu}\right]+\frac{e^{2}}{k^{2}} \operatorname{Tr}\left[\not \psi_{2} k_{2} \not p_{1} k_{1}\right] \\
& -\not p_{2} \gamma^{\mu} \gamma_{\mu} \not p_{1}+\not p_{2} \gamma^{\mu}\left\{\not p_{1}, \gamma_{\mu}\right\} \\
& =-4 \not p_{2} \not \phi_{1}+2 \not p_{2} \not p_{1}=-2 \not p_{2} \not p_{1}
\end{aligned}
$$

## The Result

- Contracting the Lorentz indices gives,

$$
\begin{aligned}
&=e^{2} \operatorname{Tr} {\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma^{\nu}\right]\left(-g_{\mu \nu}+\frac{k_{\mu} k_{\nu}}{k^{2}}\right) } \\
&=-e^{2} \operatorname{Tr} {\left[\not p_{2} \gamma^{\mu} \not p_{1} \gamma_{\mu}\right]+\frac{e^{2}}{k^{2}} \operatorname{Tr}\left[\not p_{2} \not k_{2} p_{1} \not k_{1}\right] } \\
&-\not{ }_{2} \gamma^{\mu} \gamma_{\mu} \not p_{1}+\not p_{2} \gamma^{\mu}\left\{\not p_{1}, \gamma_{\mu}\right\} \\
&=-4 \not p_{2} \not p_{1}+2 \not p_{2} \not p_{1}=-2 \not p_{2} \not p_{1} \\
&=2 e^{2} \operatorname{Tr} {\left[\not p_{2} \not p_{1}\right]=2 e^{2} 4\left(p_{1} \cdot p_{2}\right)=2 e^{2} 2 s }
\end{aligned}
$$

## The Cross Section

- We can now insert our result for the amplitude squared into the formula for the cross section,

$$
\sum|A|^{2}=2 e^{2} 2 s
$$

- This is given in this case by,

$$
\frac{1}{2 s} \frac{1}{2^{2}} \frac{1}{N_{c}^{2}} N_{C} \int d \Pi_{1} \sum|A|^{2}
$$

- The phase space integral is given by,

$$
\int d \Pi_{1}=\frac{2 \pi}{s} \delta\left(1-\frac{k^{2}}{s}\right)
$$

## The Final Result

- Putting all of this together gives us,

$$
\sigma=\frac{1}{s} \frac{4 \pi^{2} \alpha}{3} \quad e^{2}=4 \pi \alpha
$$

- If $p_{1}$ and $p_{2}$ had come from two protons with momenta $P_{1}$ and $P_{2}$ then we would have to multiply this by the PDF's and the hard cross section would be written as,

$$
\hat{\sigma}=\frac{1}{x_{1} x_{2} S} \frac{4 \pi^{2} \alpha}{3}
$$

- Where $S$ is the Mandelstam variable s for the incoming protons and $x_{1}$ and $x_{2}$ are the momentum fractions of the quarks.


## Higher-Order Effects

- We have set up techniques for computing an observable as a perturbative series,

$$
O=\alpha_{S} C_{1}+\alpha_{S}^{2} C_{2}+\alpha_{S}^{3} C_{3}+\ldots
$$

- So far we have looked at tree-level computations.
- Let us now look at one-loop amplitudes.
- One-loop level is the first time that we have quantum corrections.


## One-loop integrals

- We will start with a generic one-loop graph in $\Phi^{3}$ theory,

$$
\mathcal{L}=\frac{1}{2}\left(\partial_{\mu} \phi\right)^{2}-\frac{1}{2} m^{2} \phi^{2}-\frac{\lambda}{3!} \phi^{3}
$$

- The Feynman rules are,



## One-loop integrals

- Compute the self-energy diagram,

- Using the Feynman rules this gives,

$$
=\lambda^{2} \int \frac{d^{4} p}{(2 \pi)^{4}} \frac{1}{p^{2}-m^{2}+i \epsilon} \frac{1}{(p+k)^{2}-m^{2}+i \epsilon}
$$

- There is still a 4D integral to perform.
- This integral measure can be rewritten as

$$
\int_{0}^{\infty}|\vec{p}|^{3} d|\vec{p}| \int d \Omega_{3}
$$

## UV Divergences

- Examining the integral in the limit that $|\vec{p}|$ becomes large we see that we have,

$$
\int d \Omega_{3} \int_{0}^{\infty} \frac{|\vec{p}|^{3} d|\vec{p}|}{|\vec{p}|^{4}}=C \int_{0}^{\infty} \frac{d|\vec{p}|}{|\vec{p}|} \rightarrow \ln (\infty)
$$

- This is an example of a UV divergence.
- To deal with this divergence we will need to regulate it.
- First we could consider a cut-off regulator,

$$
\int_{0}^{\Lambda} \frac{d|\vec{p}|}{|\vec{p}|} \rightarrow \ln (\Lambda)+\text { finite }
$$

## Regulators

- The problem with this regulator is that it brakes Lorentz Invariance.
- A more ideal regulator would preserve this.
- The most common regulator is known as Dimensional regularisation.
- We perform the integral in $D \neq 4$ dimensions,

$$
\int d^{4} p \rightarrow \int d^{D} p
$$

## Dimensional Regularisation

- The integral is regulated by performing it in a dimension in which it is finite.
- The regulation parameter will then be the difference in dimension from the usual 4.
- Typically we write the dimensional as $\mathrm{D}=4-\varepsilon$.
- The integral then becomes,

$$
\int d \Omega_{3-\epsilon} \int_{0}^{\infty} \frac{|\vec{p}|^{3-\epsilon} d|\vec{p}|}{|\vec{p}|^{4}}=C \int_{a}^{\infty} d|\vec{p}||\vec{p}|^{-1-\epsilon}=\frac{1}{\epsilon} a^{-\epsilon}=\frac{1}{\epsilon} e^{-\epsilon \ln a}=\frac{1}{\epsilon}-\ln a+\ldots
$$

## Renormalisation

$$
C \int_{a}^{\infty} d|\vec{p}||\vec{p}|^{-1-\epsilon}=\frac{1}{\epsilon} a^{-\epsilon}=\frac{1}{\epsilon} e^{-\epsilon \ln a}=\frac{1}{\epsilon}-\ln a+\ldots
$$

- The divergence now appears as a pole in $\varepsilon$.
- When performing a higher-order computation we compute all the pieces and then set the regulator to zero at the end.
- The final result should be independent of $\varepsilon$.
- At the moment we still have an $\varepsilon$ dependance.
- To remove this divergence we will need to renormalise our result. We will do this in the next lecture.


## QCD Computations

- Compute QCD amplitudes in a similar way to QED.
- We compute a partonic cross section to some order in $\alpha_{\mathrm{s}}$ using Feynman diagram techniques.
- The incoming legs are then convoluted with universal PDF functions, which relate the incoming legs to the incoming particles.
- Use some experimental data to fit the PDF's in the first place.


## Summary

- Learnt about the Factorisation theorem and how this allows us to compute QCD amplitudes.
- Gone through a simple example, including computing the cross section.
- Asymptotic Freedom.
- Investigated the properties of higher order quantum predictions.

