BND school 2016

Quantum Chromodynamics

Problem 1: Gauge invariance of the QCD Lagrangian

Consider the Lagrangian describing a Dirac field ψ

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi\,.$$

We assume that ψ is a vector with three components (i.e., each of these three components is a four-component Dirac field),

$$\psi^T = (\psi_1, \psi_2, \psi_3)\,,$$

1. Show that $\mathcal{L}$ is invariant under *global* $SU(3)$ transformations, i.e. under the transformation $\psi_j \to U_{jj'}\psi_{j'}$, where $U = \exp\left[-ig_s\,\varepsilon_a T^a\right]$ is an $SU(3)$ matrix, and the generators of $SU(3)$ are denoted by T^a, and ε_a are *constants*.

2. What happens if we consider *local* $SU(3)$ transformations, i.e., we promote the ε_a to be functions of the space-time coordinates, $\varepsilon_a \to \varepsilon_a(x)$?

3. Replace the space-time derivative by a covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu - ig_s\,T^a\,G^a_\mu(x)\,,$$

where $G^a_\mu(x)$ is the *gauge field* (which corresponds to the gluon field in QCD). Determine the tranformation law for the gauge field under local transformations such that the Lagrangian is invariant under local (=gauge) transformations.

4. The *field strength tensor* of a gauge field is defined by (cf. the definition of the field strength in electrodynamics)

$$F^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s\,f^{abc}\,G^b_\mu\,G^c_\nu\,,$$

where f^{abc} are the structure constants of $SU(3)$.

   (a) Why is the term proportional to f^{abc} absent in electrodynamics? What are its implications?

   (b) Show that $F^a_{\mu\nu}$ is the commutator of two covariant derivatives,

$$F^a_{\mu\nu}\,T^a = \frac{i}{g_s}\,[D_\mu, D_\nu]\,.$$

   (c) Determine the transformation of $F^a_{\mu\nu}$ under gauge transformations.

   (d) Conclude that $F^a_{\mu\nu}F^{\mu\nu}_a$ is gauge-invariant. We can therefore add it to the QCD Lagrangian,

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \bar{\psi}(iD\!\!\!\!/ - m)\psi\,.$$

5. Are there other gauge invariant operators of dimension four and made of G^a_μ and ψ that we could add to the Lagrangian?

Problem 2: The Drell-Yan cross section at leading order

Consider the process $p\,p \to \gamma^* \to \mu^+\mu^-$ at a proton-proton collider with CoM energy S. Our goal is to compute the leading-order invariant-mass spectrum $\frac{d\sigma}{dQ^2}$, where $Q^2 = m_{\mu\mu}^2$ is the invariant mass of the lepton pair.

1. **The partonic cross sections:**

   (a) Identify all the partonic subprocesses that contribute at LO.

   (b) Write down the Feynman diagrams and the matrix elements for the subprocesses.

   (c) Perform all the phase space integrals, and show that all partonic cross sections can be written in the form

$$\hat{\sigma}_{q\bar{q}} = \frac{8\pi^2\,\alpha\,e_q^2}{N_c\,Q^2}\,\delta(1-z)\,,$$

   where α is the electromagnetic fine structure constant, e_q is the charge of the quark and $N_c = 3$ is the number of colours. The variable z is defined as $z = Q^2/\hat{s}$, where $\hat{s}$ is the partonic CoM energy.

2. **Convolution with the PDFs:**

   (a) Show that the QCD factorisation can be written in two equivalent ways

$$\frac{d\sigma}{dQ^2} = \sum_{i,j}\int_0^1 dx_1\,dx_2\,f_i(x_1)\,f_j(x_2)\,\frac{d\hat{\sigma}_{ij}}{dQ^2} = \tau\sum_{i,j}\left[\mathcal{L}_{ij}(z)\otimes\frac{1}{z}\frac{d\hat{\sigma}_{ij}}{dQ^2}\right](\tau)\,,$$

   where $\tau = Q^2/S$ and $\mathcal{L}_{ij}(z) = [f_i(x)\otimes f_j(x)](z)$ is the partonic luminosity, and the convolution is defined by

$$[f(z)\otimes g(z)](\tau) = \int_\tau^1 \frac{dz}{z}\,f(z)\,g\left(\frac{\tau}{z}\right)\,.$$

   (b) Show that

$$\frac{d\sigma}{dQ^2} = \frac{16\pi^2\,\alpha}{N_c\,S}\sum_f e_f^2\,\mathcal{L}_{q_f\bar{q}_f}(\tau)\,.$$

Problem 3: The Higgs mechanism

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \partial_\mu\Phi^\dagger\partial^\mu\Phi - V(\Phi^\dagger\Phi)\,,$$

where $V(x) = \mu^2 x + \lambda x^2$, and Φ transforms under $SU(2)\times U(1)$ gauge transformations as a doublet with hypercharge 1/2 (hypercharge is the quantum number associated to the $U(1)$ subgroup).

1. Discuss as a function of the sign of μ^2 the position of the minimum of the potential.

2. Show that one can find a gauge in which Φ can be written as

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ H(x) + v \end{pmatrix} ,$$

   where $\langle H \rangle = 0$. This gauge is called unitary gauge.

3. Show that in unitary gauge $\mathcal{L}$ is equivalent to a Lagrangian for the field H, a massless vector field A_μ, a real massive vector Z_μ and complex massive vector W_μ. Compare the number of degrees of freedom.

Problem 4: Gluon-fusion in the large-m_t limit

1. Work out the Feynman diagrams that contribute to gluon fusion at leading order. Consider all quarks massless except for the top quark.

2. Apply the 'Feynman trick',

$$\frac{1}{ABC} = \int_0^1 dx \int_0^{1-x} dy \frac{1}{[Ax + By + c(1-x-y)]^3} ,$$

   and perform the loop integration using the formulas

$$\begin{aligned} \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^n} &= \frac{(-1)^n\, i}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \Delta^{\frac{d}{2}-n} , \\ \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu\, \ell^\nu}{(\ell^2 - \Delta)^n} &= \frac{(-1)^{n-1}\, i}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-d/2-1)}{\Gamma(n)} \Delta^{1+\frac{d}{2}-n} , \\ \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu}{(\ell^2 - \Delta)^n} &= \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\mu\, \ell^\nu\, \ell^\rho}{(\ell^2 - \Delta)^n} = 0 . \end{aligned}$$

3. Take the limit where the top quark is infinitely heavy. Interpret the result.